ABELIAN CHERN-SIMONS THEORY

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Abstract. We give a construction of the abelian Chern-Simons gauge theory from the point of view of a $2+1$ dimensional topological quantum field theory. The definition of the quantum theory relies on geometric quantization ideas which have been previously explored in connection to the nonabelian Chern-Simons theory [JW, ADW]. We formulate the topological quantum field theory in terms of the category of extended 2- and 3-manifolds introduced by Walker [Wa] and prove that it satisfies the axioms of unitary topological quantum field theories formulated by Atiyah [A1].

Contents

1. Introduction

A $2+1$ dimensional topological quantum field theory (TQFT) with Lagrangian the nonabelian Chern-Simons invariant of connections was introduced by Witten several years ago [Wi1]. It was shown that this theory generates invariants of framed oriented links in arbitrary 3-manifolds. It also leads to topological invariants of closed oriented 3-manifolds endowed with a 2-framing [A3]. Witten's approach [Wi1] to defining these invariants is based on Feynman's path integral, results from two-dimensional conformal field theory and the axioms of TQFT formulated in [A1]. A mathematically rigorous derivation of Witten's 3-manifold invariants is given in [RT, Wa]. Moreover, Walker [Wa] gives a complete construction of the $SU(2)$ Chern-Simons theory, proving that it satisfies the axioms of a TQFT. A simpler model of a $2+1$ dimensional TQFT is the Chern-Simons theory with finite gauge group treated in full detail in [FQ].

In line with the above mentioned developments relating to the Chern-Simons gauge theory, this paper presents a construction of the *abelian* Chern-Simons theory as a $2+1$ dimensional TQFT. Given the group $\mathbb{T}$ of complex numbers of unit modulus and an (even) integer k, called the level, we aim to associate to every closed oriented 2-dimensional manifold Σ a finite dimensional vector space $\mathcal{H}(\Sigma)$ and to every compact oriented 3-manifold X a vector Z_X in the vector space $\mathcal{H}(\partial X)$ functorially associated to the boundary ∂X of X.

The first assignment, $\Sigma \to \mathcal{H}(\Sigma)$, results from the geometric quantization of the moduli space $\mathcal{M}_\Sigma$ of flat $\mathbb{T}$-connections on Σ, having in view that $\mathcal{M}_\Sigma$ is identified to the symplectic torus $H^1(\Sigma;\mathbb{R})/H^1(\Sigma;\mathbb{Z})$ with standard symplectic form ω_Σ. There are various models for the quantization of $\mathcal{M}_\Sigma$ according to the type of polarization one chooses on this space. The holomorphic (Kähler) quantization of $\mathcal{M}_\Sigma$ is discussed in [ADW, A2, G], where it is shown that the vector spaces constructed for various choices of complex structures on $\mathcal{M}_\Sigma$ are projectively identified. In this paper we choose to quantize $\mathcal{M}_\Sigma$ by using *real* polarizations. To each rational Lagrangian subspace L in $H^1(\Sigma;\mathbb{R})$, i.e. a Lagrangian subspace with the property that $L \cap H^1(\Sigma;\mathbb{Z})$ generates L as a vector space, there corresponds

an invariant real polarization $\mathcal{P}_L$ of the torus $\mathcal{M}_\Sigma \cong H^1(\Sigma;\mathbb{R})/H^1(\Sigma;\mathbb{Z})$. The quantization of $\mathcal{M}_\Sigma$, at level k, in this real polarization constructs a k^g-dimensional inner product space $\mathcal{H}(\Sigma, L)$, where g denotes the genus of Σ. The quantization of symplectic tori in a real polarization is discussed in detail in [Ma] and the results obtained there are directly applicable to the case of the moduli space $\mathcal{M}_\Sigma$. According to the results from [Ma], the vector spaces associated to Σ and different choices of rational Lagrangian subspaces in $H^1(\Sigma;\mathbb{R})$ are all projectively identified, similarly to the holomorphic quantization case. The projective factor, an 8-th root of unity, is expressible in terms of the Maslov-Kashiwara index $\tau(\cdot,\cdot,\cdot)$ of a triple of Lagrangian subspaces of $H^1(\Sigma;\mathbb{R})$. More precisely, for any two rational Lagrangian subspaces $L_1, L_2 \subset H^1(\Sigma;\mathbb{R})$ there is a canonically defined unitary operator $F_{L_2L_1} : \mathcal{H}(\Sigma, L_1) \to \mathcal{H}(\Sigma, L_2)$; if L_3 is another such Lagrangian subspace in $H^1(\Sigma;\mathbb{R})$, then the unitary operators relating the vector spaces corresponding to Σ and each of these Lagrangian subspaces satisfy the composition law: $F_{L_1L_3} \circ F_{L_3L_2} \circ F_{L_2L_1} = \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}\tau(L_1,L_2,L_3)}\, I$. The idea of using real polarizations in quantizing the moduli space $\mathcal{M}_\Sigma$ and the method of construction of the vector space $\mathcal{H}(\Sigma, L)$ was inspired by [JW], where, in the context of the nonabelian Chern-Simons theory, the authors address the problem of quantizing the moduli space of flat $SU(2)$ connections on a closed oriented 2-manifold in a particular real polarization of that space.

For the second assignment, $X \to Z_X$, we start from the known fact that, for X a compact oriented 3-manifold with boundary, the image of the restriction map $H^1(X;\mathbb{R}) \to H^1(\partial X;\mathbb{R})$ defines a Lagrangian subspace L_X in the symplectic vector space $(H^1(\partial X;\mathbb{R}), \omega_{\partial X})$. Then we provide a canonical construction which defines Z_X as an element of the vector space $\mathcal{H}(\partial X, L_X)$ associated to the closed 2-manifold ∂X and the rational Lagrangian subspace L_X. The construction of Z_X relies on geometric quantization ideas. Following the Chern-Simons line construction exposed in [F], we introduce a prequantum line bundle $\mathcal{L}_{\partial X}$ over the moduli space of flat connections $\mathcal{M}_{\partial X}$. The line bundle $\mathcal{L}_{\partial X}$ carries a natural connection with curvature the symplectic form $k\omega_{\partial X}$. The rational Lagrangian

subspace $L_X \subset H^1(\partial X;\mathbb{R})$ defines an invariant real polarization on $\mathcal{M}_{\partial X}$. One of the leaves of this polarization is the image Λ_X in $\mathcal{M}_{\partial X}$ of the moduli space $\mathcal{M}_X$ of flat $\mathbb{T}$-connections on X under the map $r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}$ determined by restricting connections on X to the boundary ∂X. We show that the leave Λ_X is a Bohr-Sommerfeld leave, that is, the restriction of the line bundle $\mathcal{L}_{\partial X}$ to Λ_X admits nonzero covariantly constant global sections. The vector Z_X is defined as a section of the line bundle $\mathcal{L}_{\partial X}\big|_{\Lambda_X}$ times a section of the bundle of half-densities on Λ_X. The former section is defined in terms of the Chern-Simons functional of flat connections on X and the latter in terms of the Reidemeister torsion invariant of the 3-manifold X. For X a closed manifold the construction gives for Z_X a complex number. The definition of Z_X, that is, finding the appropriate ingredients which should enter into this definition, was inspired in part by the results for the nonabelian Chern-Simons theory from [JW], as well as by the path integral formulation of the closed 3-manifold $SU(2)$ invariant given in [FG, Wi1].

We find that a natural way to incorporate the assignments $\Sigma \to \mathcal{H}(\Sigma)$ and $X \to Z_X$, defined as outlined above, into a TQFT is to make use of the category of *extended* 2- and 3-manifolds introduced by Walker in his treatment [Wa] of the nonabelian Chern-Simons theory. In this paper an extended 2-manifold is a pair (Σ, L), with Σ a closed oriented 2-manifold and L a rational Lagrangian subspace in $H^1(\Sigma;\mathbb{R})$. An extended 3-manifold is a triple (X, L, n), with X a compact oriented 3-manifold, L a rational Lagrangian subspace in $H^1(\partial X;\mathbb{R})$ and $n \in \mathbb{Z}/8\mathbb{Z}$. Then the theory assigns to (Σ, L) the finite dimensional inner product space $\mathcal{H}(\Sigma, L)$ and to (X, L, n) the vector $Z_{(X,L,n)} = \mathrm{e}^{\frac{\pi \mathrm{i}}{4}n} F_{LL_X}(Z_X)$ belonging to the vector space $\mathcal{H}(\partial X, L)$. We prove that these assignments satisfy the axioms [A1] of a unitary TQFT, that is, the functoriality, orientation, disjoint union and gluing properties.

This paper is organized as follows. In Sect.2 we review some fairly standard material on the moduli space of flat $\mathbb{T}$-connections. We outline properties of this moduli space, essential to the constructions of the subsequent sections, for the cases when connections are on 2- and 3-dimensional manifolds. In Sect.3 we recall

first the definitions of induced principal bundles and induced connections. Then we introduce the Chern-Simons functional of $\mathbb{T}$-connections on a 3-manifold with and without boundary and present properties of this functional which are relevant for the sections to follow. In Sect.4 we give the construction of the prequantum line bundle over the moduli space of flat $\mathbb{T}$-connections on a closed 2-manifold. Then we show that, if X is a 3-manifold with boundary, the pullback of the prequantum line bundle $\mathcal{L}_{\partial X}$ over $\mathcal{M}_{\partial X}$ to $\mathcal{M}_X$ has a covariantly constant section defined in terms of the Chern-Simons functional. The constructions of this section are based almost entirely on the material in ([F],§2). Sect.5 defines the quantum theory. We construct the finite dimensional Hilbert space $\mathcal{H}(\Sigma, L)$ associated to a closed oriented 2-manifold Σ and a rational Lagrangian subspace $L \subset H^1(\Sigma; \mathbb{R})$. Then, for a compact oriented 3-manifold X, we construct the vector Z_X belonging the Hilbert space $\mathcal{H}(\partial X, L_X)$. Sect.6 contains the definition of the abelian Chern-Simons TQFT. Following [Wa] we introduce the notions of extended 2- and 3-manifolds, extended morphisms and gluing of extended 3-manifolds. Using the results of Sect. 5, we define then a $2+1$ dimensional TQFT based on this category of extended manifolds and prove that it satisfies the required axioms. Sect.7 relates the definition of the vector Z_X associated to the 3-manifold X, given in Sect.5, to results obtained from the path integral approach to the Chern-Simons gauge theory.

Throughout this paper manifolds, bundles, sections, maps are assumed smooth.

2. The moduli space of flat $\mathbb{T}$-connections

2.1. **The space of $\mathbb{T}$-connections.** Let M be a smooth manifold. The space $\mathcal{G}_M = \mathrm{Map}(M, \mathbb{T})$ forms a group under pointwise multiplication. $\mathcal{G}_M$ is the *group of gauge transformations* on M. The space of components $\pi_0(\mathcal{G}_M)$ is isomorphic to $H^1(M; \mathbb{Z})$ [AB]. For any principal $\mathbb{T}$-bundle $\pi : P \to M$, the group $\mathcal{G}_M$ can be identified with the group $\mathcal{G}_P = \mathrm{Aut}(P)$ of bundle automorphisms of P, that is, the group of $\mathbb{T}$-equivariant maps $\phi : P \to P$ covering the identity on M. An element u of $\mathcal{G}_M$ defines a bundle map $\phi_u : P \to P$ by $\phi(p) = p \cdot u(\pi(p))$ and, conversely,

an automorphism $\phi : P \to P$ defines a map $u_\phi : M \to \mathbb{T}$. We call $\mathcal{G}_P$ the group of gauge transformations of P.

Let $\mathcal{A}_M$ denote the space of $\mathbb{T}$-connections on M. An element Θ in $\mathcal{A}_M$ is a connection on a principal $\mathbb{T}$-bundle $P \xrightarrow{\pi} M$. Thus $\mathcal{A}_M$ is equal to the union

$$\mathcal{A}_M = \bigsqcup_P \mathcal{A}_P$$

over all principal $\mathbb{T}$-bundles P on M. For each bundle P the space $\mathcal{A}_P$ of connections on P is an affine space with vector space $2\pi \mathrm{i}\,\Omega^1(M;\mathbb{R})$, where $\Omega^1(M;\mathbb{R})$ is the space of 1-forms on M (the Lie algebra of $\mathbb{T}$ is identified with $2\pi\mathrm{i}\,\mathbb{R}$).

A bundle isomorphism between two principal $\mathbb{T}$-bundles P and P' over M is a $\mathbb{T}$-equivariant map $\phi : P' \to P$ which covers the identity on M. Two elements Θ and Θ' in $\mathcal{A}_M$ are called *gauge equivalent* if there exists an isomorphism $\phi : P' \to P$ such that $\Theta' = \phi^*\Theta$, that is, the connection Θ' on P' is the pullback under ϕ of the connection Θ on P. This defines an equivalence relation $\Theta' \sim \Theta$ on $\mathcal{A}_M$. We let $\mathcal{A}_M/\!\sim$ denote the space of gauge equivalence classes of $\mathbb{T}$-connections on M.

For each $\mathbb{T}$-bundle P there is a right-action of $\mathcal{G}_M$ on the space of connections $\mathcal{A}_P$. If $u : M \to \mathbb{T}$ is an element of $\mathcal{G}_M$ with associated bundle automorphism $\phi_u : P \to P$, the action of u on $\Theta \in \mathcal{A}_P$ is described by

$$\Theta \cdot u \;=\; \phi_u^*\Theta \;=\; \Theta + (u\cdot\pi)^*\vartheta\,, \tag{2.1}$$

where ϑ is the Maurer-Cartan form of $\mathbb{T}$, that is, $\frac{\vartheta}{2\pi\mathrm{i}}$ generates $H^1(\mathbb{T};\mathbb{Z}) \subset H^1(\mathbb{T};\mathbb{R})$.

A principal $\mathbb{T}$-bundle $P \to M$ has flat connections if and only if $c_1(P) \in \operatorname{Tors} H^2(M;\mathbb{Z})$, that is, its first Chern class is a torsion class. The subspace $\mathcal{A}^f_M \subset \mathcal{A}_M$ of flat $\mathbb{T}$-connections on M is given by the union

$$\mathcal{A}^f_M = \bigsqcup_{\substack{P \\ c_1(P)\in \operatorname{Tors} H^2(M;\mathbb{Z})}} \mathcal{A}^f_P\,,$$

where $\mathcal{A}^f_P$ is the space of connections Θ on P with curvature $F_\Theta = d\Theta = 0$. For each $\mathbb{T}$-bundle P with first Chern class torsion, the *moduli space of flat connections* on P is the quotient space $\mathcal{M}_P = \mathcal{A}^f_P/\mathcal{G}_P$.

If P and P' are $\mathbb{T}$-bundles over M with $c_1(P) = c_1(P') \in \operatorname{Tors} H^2(M;\mathbb{Z})$, then there is a canonical isomorphism $\mathcal{M}_P \cong \mathcal{M}_{P'}$. To see this choose a bundle isomorphism $\phi : P' \to P$. The bundle map ϕ determines an isomorphism $\phi^* : \mathcal{A}_P \to \mathcal{A}_{P'}$ given by the pullback of a connection Θ in $\mathcal{A}_P$ to a connection $\phi^*\Theta$ in $\mathcal{A}_{P'}$. The map ϕ^* pushes down to an isomorphism between the quotients by the groups of gauge transformations $\phi^* : \mathcal{A}_P/\mathcal{G}_P \to \mathcal{A}_{P'}/\mathcal{G}_{P'}$. The induced isomorphism $\mathcal{A}_P/\mathcal{G}_P \cong \mathcal{A}_{P'}/\mathcal{G}_{P'}$ does not depend on the choice of bundle isomorphism $\phi : P' \to P$.

For each torsion class $p \in \operatorname{Tors} H^2(M;\mathbb{Z})$ let

$$\mathcal{A}^f_{M,p} = \bigsqcup_{\substack{P \\ c_1(P)=p}} \mathcal{A}^f_P$$

The space $\mathcal{M}_{M,p} = \mathcal{A}^f_{M,p}/\sim$ is the moduli space of flat connections on principal $\mathbb{T}$-bundles over M with first Chern class equal to p. For any principal $\mathbb{T}$-bundle $P \to M$ with $c_1(P) = p$, there is a natural isomorphism $\mathcal{M}_{M,p} \cong \mathcal{M}_P$. The moduli space $\mathcal{M}_M = \mathcal{A}^f_M/\sim$ of gauge equivalence classes of flat $\mathbb{T}$-connections on M is equal to the disjoint union

$$\mathcal{M}_M \;=\; \bigsqcup_{p \in \operatorname{Tors} H^2(M;\mathbb{Z})} \mathcal{M}_{M,p}$$

and we have

Proposition 2.2. *Let M be a smooth manifold.*
(i) There is a natural identification

$$\mathcal{M}_M \;=\; H^1(M;\mathbb{T})$$

(ii) $\pi_0(\mathcal{M}_M) \cong \operatorname{Tors} H^2(M;\mathbb{Z})$ and each connected component of $\mathcal{M}_M$ is diffeomorphic to the torus $H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})$.

Proof. The first assertion is a consequence of a standard result in the theory of connections ([KN], ch.II) which provides the natural identification

$$\mathcal{M}_M = \prod_\alpha \operatorname{Hom}(\pi_1(M_\alpha, *), \mathbb{T}),$$

where the index α labels the connected components M_α of M. The relation between π_1 and the first homology group then gives (i). For (ii), consider the exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow[\exp 2\pi\mathrm{i}(\cdot)]{} \mathbb{T} \longrightarrow 1$$

and the induced exact cohomology sequence

$$\begin{aligned} &0 \to H^1(M;\mathbb{Z}) \to H^1(M;\mathbb{R}) \to \\ &\qquad \to H^1(M;\mathbb{T}) \xrightarrow{\delta} H^2(M;\mathbb{Z}) \xrightarrow{i} H^2(M;\mathbb{R}) \to \cdots \end{aligned} \tag{2.3}$$

All the maps in the above sequence are group homomorphisms and $\operatorname{Im}\delta = \operatorname{Ker} i = \{p \mid p \in \operatorname{Tors} H^2(M;\mathbb{Z})\}$. Thus, for each torsion class p in $H^2(M;\mathbb{Z})$, we have $\mathcal{M}_{M,p} = \delta^{-1}(p) \cong \delta^{-1}(0) \cong H^1(M;\mathbb{R})/H^1(M;\mathbb{Z})$. □

2.2. **The moduli space of flat $\mathbb{T}$-connections on a 2-manifold.** Let Σ be a closed oriented 2-dimensional manifold. We note that a principal $\mathbb{T}$-bundle $Q \to \Sigma$ has flat connections if and only if $c_1(Q) = 0$, that is, the bundle Q is trivializable. For Σ the exact sequence (2.3) splits to

$$0 \longrightarrow H^1(\Sigma;\mathbb{Z}) \longrightarrow H^1(\Sigma;\mathbb{R}) \longrightarrow H^1(\Sigma;\mathbb{T}) \longrightarrow 0$$

Thus the moduli space of flat $\mathbb{T}$-connections on Σ is the torus

$$\mathcal{M}_\Sigma \ = \ H^1(\Sigma;\mathbb{T}) \ \cong \ H^1(\Sigma,\mathbb{R})/H^1(\Sigma,\mathbb{Z})\,.$$

$\mathcal{M}_\Sigma$ carries a natural symplectic structure. The space $H^1(\Sigma;\mathbb{R})$ is a symplectic vector space with symplectic form ω_Σ defined by cup product followed by evaluation on the fundamental cycle, or in de Rham cohomology by

$$\omega_\Sigma([\alpha],[\beta]) \ = \int\limits_\Sigma \alpha\wedge\beta\,, \qquad [\alpha],[\beta]\in H^1(\Sigma;\mathbb{R})\,.$$

The integer lattice $H^1(\Sigma;\mathbb{Z}) \subset H^1(\Sigma;\mathbb{R})$ is self-dual with respect to ω_Σ. The symplectic form ω_Σ on $H^1(\Sigma;\mathbb{R})$ descends to a symplectic form ω_Σ on the quotient

torus $\mathcal{M}_\Sigma \cong H^1(\Sigma;\mathbb{R})/H^1(\Sigma;\mathbb{Z})$. We note that since the tangent space $T\mathcal{M}_\Sigma$ is identified with $2\pi\mathrm{i}\, H^1(\Sigma;\mathbb{R})$ we have

$$\omega_\Sigma([\dot\eta],[\dot\eta']) \;=\; -\frac{1}{4\pi^2}\int\limits_\Sigma \dot\eta\wedge\dot\eta'\,, \tag{2.4}$$

for any $[\dot\eta],[\dot\eta'] \in T\mathcal{M}_\Sigma$. Moreover, ω_Σ is normalized so that it gives $\mathcal{M}_\Sigma$ total volume equal to 1, that is,

$$\int\limits_{\mathcal{M}_\Sigma} \frac{(\omega_\Sigma)^g}{g!} \;=\; 1\,,$$

where $g = \frac{1}{2}\dim H^1(\Sigma;\mathbb{R})$. If $-\Sigma$ denotes the manifold Σ with the opposite orientation then by $H^1(-\Sigma;\mathbb{R})$ we understand the symplectic vector space $H^1(\Sigma;\mathbb{R})$ with symplectic form $-\omega_\Sigma$.

2.3. **The moduli space of flat $\mathbb{T}$-connections on a 3-manifold.** Let X be a compact oriented 3-manifold with boundary $\partial X \neq \emptyset$. The restriction of a connection over X to the boundary ∂X induces a map $r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}$ from the moduli space of flat $\mathbb{T}$-connections on X into the moduli space of flat $\mathbb{T}$-connections on ∂X. In view of the identifications $\mathcal{M}_X = H^1(X;\mathbb{T})$ and $\mathcal{M}_{\partial X} = H^1(\partial X;\mathbb{T})$, the moduli spaces $\mathcal{M}_X$ and $\mathcal{M}_{\partial X}$ are compact abelian Lie groups and the restriction map r_X is a group homomorphism. We have

Proposition 2.5. *(i) The image $\Lambda_X = \mathrm{Im}\,\{r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}\}$ is a Lagrangian submanifold of the symplectic torus $(\mathcal{M}_{\partial X}, \omega_{\partial X})$.*
(ii) $r_X : \mathcal{M}_X \to \Lambda_X$ is a principal fibre bundle with structure group a compact abelian Lie group H which fits into the exact sequence

$$1 \longrightarrow \mathbb{T}^q \longrightarrow H \longrightarrow \mathrm{Tors}\, H^2(X;\mathbb{Z}) \longrightarrow 0$$

with $q = \dim H^1(X,\partial X;\mathbb{R}) - \dim H^0(\partial X;\mathbb{R}) + \dim H^0(X;\mathbb{R}) - \dim H^0(X,\partial X;\mathbb{R})$.

Proof. There is a long exact cohomology sequence ([Br],ch.V) associated to the pair of spaces $\partial X \subset X$

$$\cdots \to H^i(X,\partial X;G) \to H^i(X;G) \to H^i(\partial X;G) \to H^{i+1}(X,\partial X;G) \to \cdots , \tag{2.6}$$

where $G = \mathbb{Z}, \mathbb{R}$ or $\mathbb{T}$. At the same time the exact sequence of abelian groups $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 1$ induces the long exact cohomology sequence ([Br], ch.V)

$$\cdots \longrightarrow H^i(A;\mathbb{Z}) \longrightarrow H^i(A;\mathbb{R}) \longrightarrow H^i(A;\mathbb{T}) \longrightarrow H^{i+1}(A;\mathbb{Z}) \longrightarrow \cdots , \tag{2.7}$$

where A stands for either the space ∂X or X, or $(X,\partial X)$ for the relative cohomology. The above exact sequences fit into a commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & & H^1(X,\partial X;\mathbb{Z}) & \longrightarrow & H^1(X;\mathbb{Z}) & \longrightarrow & H^1(\partial X;\mathbb{Z}) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & & H^1(X,\partial X;\mathbb{R}) & \longrightarrow & H^1(X;\mathbb{R}) & \xrightarrow{\dot r_X} & H^1(\partial X;\mathbb{R}) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & & H^1(X,\partial X;\mathbb{T}) & \longrightarrow & H^1(X;\mathbb{T}) & \xrightarrow{r_X} & H^1(\partial X;\mathbb{T}) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & & H^2(X,\partial X;\mathbb{Z}) & \longrightarrow & H^2(X;\mathbb{Z}) & \longrightarrow & H^2(\partial X;\mathbb{Z}) & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & \vdots & & \vdots & & \vdots &
\end{array}$$

where all the maps are group homomorphisms. The differentiable and group structure of $\mathbb{T}$ make

$$\begin{aligned}
&1 \to H^0(X,\partial X;\mathbb{T}) \to H^0(X;\mathbb{T}) \to H^0(\partial X;\mathbb{T}) \to \\
&\qquad \to H^1(X,\partial X;\mathbb{T}) \to H^1(X;\mathbb{T}) \xrightarrow{r_X} \underset{H^1(\partial X;\mathbb{T})}{\underset{\cap}{\Lambda_X}} \to 1
\end{aligned} \tag{2.8}$$

an exact sequence of homomorphisms of compact abelian Lie groups. Λ_X is a compact subgroup and, therefore, a submanifold of $\mathcal{M}_{\partial X} = H^1(\partial X;\mathbb{T})$. The

relevant part of the sequence (2.7) for $A = (X, \partial X)$ is

$$\cdots \to H^1(X, \partial X; \mathbb{T}) \to H^2(X, \partial X; \mathbb{Z}) \to H^2(X, \partial X; \mathbb{R}) \to \cdots$$

which implies that $\pi_0(H^1(X, \partial X; \mathbb{T})) \cong \operatorname{Tors} H^2(X, \partial X; \mathbb{Z})$. The universal coefficient theorem (U.C.T.) ([Br],ch.V) together with Poincaré duality (P.D.) give the isomorphisms

$$\operatorname{Tors} H^2(X, \partial X; \mathbb{Z}) \underset{\text{(U.C.T.)}}{\cong} \operatorname{Tors} H_1(X, \partial X; \mathbb{Z}) \underset{\text{(P.D.)}}{\cong} \operatorname{Tors} H^2(X; \mathbb{Z}) \tag{2.9}$$

Thus (ii) follows from (2.8) and (2.9). To prove (i), first recall that the tangent spaces to $\mathcal{M}_X$ and $\mathcal{M}_{\partial X}$ are identified with the corresponding real cohomology groups by $T\mathcal{M}_X = 2\pi\mathrm{i}\, H^1(X; \mathbb{R})$ and $T\mathcal{M}_{\partial X} = 2\pi\mathrm{i} H^1(\partial X; \mathbb{R})$. The differential $\dot r_X$ of r_X maps $T\mathcal{M}_X$ onto $T\Lambda_X \subset T\mathcal{M}_{\partial X}$ and we have

$$\begin{aligned} \omega_{\partial X}(\dot r_X[\dot\Theta], \dot r_X[\dot\Theta']) \;&=\; -\frac{1}{4\pi^2} \int\limits_{\partial X} \dot r_X(\dot\Theta) \wedge \dot r_X(\dot\Theta') \\ &=\; -\frac{1}{4\pi^2} \int\limits_{X} d\,(\dot\Theta \wedge \dot\Theta') \;=\; 0\,, \quad \text{for any } \; [\dot\Theta], [\dot\Theta'] \in T\mathcal{M}_X\,. \end{aligned}$$

Since $\omega_{\partial X}\big|_{T\Lambda_X} \equiv 0$, Λ_X is an isotropic submanifold of $\mathcal{M}_{\partial X}$. The exact sequence (2.6) for $G = \mathbb{R}$ and the Poincaré duality isomorphism $H^{3-i}(X; \mathbb{R}) \cong H_i(X, \partial X; \mathbb{R})$ give the commutative diagram ([Br],ch.VI):

$$\begin{array}{ccccc} H^1(X;\mathbb{R}) & \xrightarrow{\dot r_X} & H^1(\partial X;\mathbb{R}) & \longrightarrow & H^2(X,\partial X;\mathbb{R}) \\ \big\downarrow \cong & & \big\downarrow \cong & & \big\downarrow \cong \\ H_2(X,\partial X;\mathbb{R}) & \longrightarrow & H_1(\partial X;\mathbb{R}) & \xrightarrow{j} & H_1(X;\mathbb{R}) \end{array}$$

where j and $\dot r_X$ are adjoint maps. Therefore $\dim \operatorname{Im} \dot r_X = \dim H^1(\partial X; \mathbb{R}) - \dim \operatorname{Ker} j$. Combining this with the exact sequence condition $\operatorname{Im} \dot r_X = \operatorname{Ker} j$, we find that $\dim \operatorname{Im} \dot r_X = \frac{1}{2} \dim H^1(\partial X; \mathbb{R})$. Therefore

$$L_X \;=\; \operatorname{Im} \{\dot r_X : H^1(X; \mathbb{R}) \longrightarrow H^1(\partial X; \mathbb{R})\}$$

is a Lagrangian subspace of $H^1(\partial X; \mathbb{R})$ and Λ_X a Lagrangian submanifold of $(\mathcal{M}_{\partial X}, \omega_{\partial X})$. $\square$

3. The Chern-Simons functional

3.1. **Induced principal bundles and induced connections.** Consider the group $SU(2)$ of complex 2×2 unitary matrices of determinant equal to 1. A maximal torus in $SU(2)$ is isomorphic to the group $\mathbb{T}$ of complex numbers of unit modulus. We let

$$\begin{aligned} \rho : \mathbb{T} &\hookrightarrow SU(2) \\ \mathrm{e}^{2\pi\mathrm{i}\varphi} &\mapsto \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\varphi} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}\varphi} \end{pmatrix} \end{aligned}$$

be the inclusion homomorphism from the circle group $\mathbb{T}$ into $SU(2)$. The corresponding inclusion of the Lie algebra $\mathrm{Lie}\,\mathbb{T} = 2\pi\mathrm{i}\,\mathbb{R}$ into the Lie algebra $\mathfrak{su}(2)$ of traceless skew-hermitian 2×2 complex matrices is

$$\begin{aligned} \rho_* : \mathrm{Lie}\,\mathbb{T} &\hookrightarrow \mathfrak{su}(2) \\ \alpha &\mapsto \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \end{aligned}$$

Let Ad denote the adjoint action of $SU(2)$ on its Lie algebra and let $\hat{\vartheta}$ denote the Maurer-Cartan form of $SU(2)$. On the Lie algebra $\mathfrak{su}(2)$ we choose the symmetric Ad-invariant bilinear form

$$\begin{aligned} \langle \cdot, \cdot \rangle^\flat \, : \, \mathfrak{su}(2) \times \mathfrak{su}(2) &\longrightarrow \mathbb{R} \\ (a\,, b) \qquad &\longmapsto \langle a, b \rangle^\flat \; = \; \frac{1}{8\pi^2}\,\mathrm{Tr}(ab) \end{aligned}$$

The bilinear form $\langle \cdot, \cdot \rangle^\flat$ is normalized so that the closed 3-form $\frac{1}{6}\langle \hat{\vartheta} \wedge [\hat{\vartheta}, \hat{\vartheta}] \rangle^\flat$ represents an integral cohomology class in $H^3(SU(2); \mathbb{R})$. The commutator $[\cdot, \cdot]$ of two $\mathfrak{su}(2)$-valued forms is defined by taking the wedge product of forms and the natural Lie bracket in the $\mathfrak{su}(2)$ Lie algebra. The bilinear form $\langle \cdot, \cdot \rangle^\flat$ on $\mathfrak{su}(2)$ restricts to a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathrm{Lie}\,\mathbb{T}$:

$$\begin{aligned} \langle \cdot, \cdot \rangle \, : \, \mathrm{Lie}\mathbb{T} \times \mathrm{Lie}\mathbb{T} &\longrightarrow \mathbb{R} \\ (\alpha, \beta) \qquad &\longmapsto \langle \alpha, \beta \rangle \; = \; \langle \rho_*\alpha, \rho_*\beta \rangle^\flat \; = \; \frac{1}{4\pi^2}\,\alpha\beta \end{aligned}$$

As in the previous section we let ϑ denote the Maurer-Cartan form on $\mathbb{T}$ and note that $\rho^*(\hat{\vartheta}) = \rho_* \vartheta$.

Let M be a smooth manifold and consider a principal $\mathbb{T}$-bundle $\pi : P \to M$. We can use the inclusion homomorphism $\rho : \mathbb{T} \hookrightarrow SU(2)$ to extend the $\mathbb{T}$-bundle P on M to a $SU(2)$-bundle $\hat{P}$ on M. The induced $SU(2)$-bundle $\hat{P}$ is defined as the quotient of $P \times SU(2)$ by the right $\mathbb{T}$-action given by $(p, a) \cdot \lambda = (p \cdot \lambda, \rho(\lambda)^{-1} a)$, for any $\lambda \in \mathbb{T}$ and $(p, a) \in P \times SU(2)$. Let $[p, a] = (p, a) \cdot \mathbb{T}$ denote the $\mathbb{T}$-orbit through the point (p, a). The natural right $SU(2)$-action on $P \times SU(2)$, $(p, a) \cdot a' = (p, aa')$, commutes with the $\mathbb{T}$-action and passes therefore to the quotient $\hat{P} = P \times_{\mathbb{T}} SU(2) = (P \times SU(2))/\mathbb{T}$. Hence, $\hat{\pi} : \hat{P} \to M$ with projection $\hat{\pi}([p, a]) = \pi(p)$ is a principal $SU(2)$-bundle. We have a natural morphism of principal bundles

$$\begin{aligned} \rho_P : P &\longrightarrow \hat{P} \\ p &\longmapsto [p, e] \end{aligned}$$

covering the identity map on M.

For any connection Θ on the $\mathbb{T}$-bundle P there is an induced connection $\hat{\Theta}$ on the $SU(2)$-bundle $\hat{P}$. The latter is determined by the $\mathfrak{su}(2)$-valued 1-form on $P \times SU(2)$:

$$\hat{\Theta}_{(p,a)} \;=\; Ad_{a^{-1}}(\rho_*\, pr_1^* \,\Theta_p) \;+\; pr_2^* \,\hat{\vartheta}_a \,, \tag{3.1}$$

where $pr_1 : P \times SU(2) \to P$ and $pr_2 : P \times SU(2) \to SU(2)$ are the natural projections. One can easily prove that $\hat{\Theta}$ is invariant under the $\mathbb{T}$-action and vanishes along the $\mathbb{T}$-orbits in $P \times SU(2)$. Thus $\hat{\Theta}$ pushes down to a $\mathfrak{su}(2)$-valued 1-form on $\hat{P} = P \times_{\mathbb{T}} SU(2)$ and defines a connection on $\hat{P}$. The pullback of $\hat{\Theta}$ under the bundle map $\rho_P : P \to \hat{P}$ is

$$\rho_P^* \hat{\Theta} \;=\; \rho_* \Theta \,.$$

Similarly, the curvature forms $F_\Theta = d\Theta$ and $F_{\hat{\Theta}} = d\hat{\Theta} + \frac{1}{2}[\hat{\Theta}, \hat{\Theta}]$ are related by

$$\rho_P^* F_{\hat{\Theta}} \;=\; \rho_* F_\Theta \,.$$

3.2. **The Chern-Simons functional for a closed 3-manifold.** Let X be a closed oriented 3-manifold. As previously shown, for any principal $\mathbb{T}$-bundle $P \to X$ there is an induced $SU(2)$-bundle $\hat{P} = P \times_{\mathbb{T}} SU(2) \to X$ and for any $\mathbb{T}$-connection Θ on P an induced $SU(2)$-connection $\hat{\Theta}$ on $\hat{P}$. Now, any principal $SU(2)$-bundle over a manifold of dimension ≤ 3 is trivializable. Thus $\hat{P}$ admits a global section and we have:

Definition 3.2. The Chern-Simons functional of a $\mathbb{T}$-connection Θ on $P \to X$ is defined by

$$S_{X,P}(\Theta) = \int\limits_X \hat{s}^*\alpha(\hat{\Theta}) \quad (\mathrm{mod}\ 1)\,,$$

where $\alpha(\hat{\Theta}) \in \Omega^3(\hat{P};\mathbb{R})$ is the Chern-Simons form of the induced $SU(2)$-connection $\hat{\Theta}$ on $\hat{P} = P \times_{\mathbb{T}} SU(2)$,

$$\alpha(\hat{\Theta}) = \langle \hat{\Theta} \wedge F_{\hat{\Theta}} \rangle^\flat - \frac{1}{6}\langle \hat{\Theta} \wedge [\hat{\Theta}, \hat{\Theta}] \rangle^\flat\,,$$

and $\hat{s} : X \to \hat{P}$ is a global section of the $SU(2)$-bundle $\hat{P}$.

We need to show that the definition of $S_{X,P}(\Theta)$ does not depend on the choice of section $\hat{s} : X \to \hat{P}$. This is a consequence of the following property of the $SU(2)$ Chern-Simons form $\alpha(\hat{\Theta})$:

Proposition 3.3. *If $\hat{s}, \hat{s}_1 : X \to \hat{P}$ are two sections of the $SU(2)$-bundle $\hat{P}$ over X, then*

$$\int\limits_X \hat{s}_1^*\alpha(\hat{\Theta}) = \int\limits_X \hat{s}^*\alpha(\hat{\Theta}) + \int\limits_X d\langle Ad_{a^{-1}}\, \hat{s}^*\hat{\Theta} \wedge a^*\hat{\vartheta} \rangle^\flat - \int\limits_X \frac{1}{6}\, a^*\langle \hat{\vartheta} \wedge [\hat{\vartheta}, \hat{\vartheta}] \rangle^\flat\,,$$

where $a : X \to SU(2)$ is the map defined by $\hat{s}_1(x) = \hat{s}(x) \cdot a(x)$, for any $x \in X$.

Proof. The expression follows after basic computations from the standard relation $\hat{s}_1^*\hat{\Theta} = Ad_{a^{-1}}(\hat{s}\,\hat{\Theta}) + a^*\hat{\vartheta}$. □

On the right-hand side of the equation in (3.3), the second integral vanishes by Stokes theorem and the assumption $\partial X = \emptyset$, while the last integral is an integer due to the normalization of the bilinear form $\langle \cdot, \cdot \rangle^\flat$ on $\mathfrak{su}(2)$. This proves that $S_{X,P}(\Theta)$ is well-defined.

Remark 3.4. If the bundle $P \to X$ is trivializable, then let $s : X \to P$ be a section and take $\hat{s} : X \to \hat{P}$ to be $\hat{s} = \rho_P \circ s$. Then we have

$$\begin{aligned} \hat{s}^*\alpha(\hat{\Theta}) &= s^*\rho_P^*\alpha(\hat{\Theta}) = s^*\langle \rho_P^*\hat{\Theta} \wedge \rho_P^* F_{\hat{\Theta}} \rangle^\flat - \frac{1}{6} s^* \langle \rho_P^*\hat{\Theta} \wedge [\rho_P^*\hat{\Theta}, \rho_P^*\hat{\Theta}] \rangle^\flat \\ &= s^*\langle \rho_*\Theta \wedge \rho_* F_\Theta \rangle^\flat = s^*\langle \Theta \wedge F_\Theta \rangle \end{aligned}$$

and the Chern-Simons functional has as expected the expression

$$S_{X,P}(\Theta) \,=\, \int_X s^*\langle \Theta \wedge F_\Theta \rangle \quad (\mathrm{mod}\ 1) \tag{3.5}$$

Theorem 3.6. *The Chern-Simons functional $S_{X,P} : \mathcal{A}_P \to \mathbb{R}/\mathbb{Z}$ defined for a closed oriented 3-manifold X and a principal $\mathbb{T}$-bundle $P \to X$ has the following properties:*

(a) Functoriality

If $\phi : P' \to P$ is a morphism of principal $\mathbb{T}$-bundles covering an orientation preserving diffeomorphism $\bar{\phi} : X' \to X$ and if Θ is a connection on P, then

$$S_{X',P'}(\phi^*\Theta) \,=\, S_{X,P}(\Theta)$$

(b) Orientation

If $-X$ denotes the manifold X with the opposite orientation, then

$$S_{-X,P}(\Theta) \,=\, -S_{X,P}(\Theta)$$

(c) Disjoint union

Let X be the disjoint union $X = X_1 \sqcup X_2$ and $P = P_1 \sqcup P_2$ a principal $\mathbb{T}$-bundle over X. If Θ_i are connections on $P_i \to X_i$, then

$$S_{X_1 \sqcup X_2, P_1 \sqcup P_2}(\Theta_1 \sqcup \Theta_2) \,=\, S_{X_1,P_1}(\Theta_1) \,+\, S_{X_2,P_2}(\Theta_2)$$

Proof. (a) To the principal $\mathbb{T}$-bundles P and P' there correspond the induced $SU(2)$-bundles $\hat{P} = P \times_{\mathbb{T}} SU(2)$ and $\hat{P}' = P' \times_{\mathbb{T}} SU(2)$, respectively. The morphism of $\mathbb{T}$-bundles $\phi : P' \to P$ induces a morphism of $SU(2)$-bundles $\hat{\phi} : \hat{P}' \to \hat{P}$

covering $\bar{\phi} : X' \to X$. It is defined by $\hat{\phi}([p', a]) = [\phi(p'), a]$. For any section $\hat{s}' : X' \to \hat{P}'$ we have a section $\hat{s} = \hat{\phi} \circ \hat{s}' \circ \bar{\phi}^{-1} : X \to \hat{P}$. Then we get

$$S_{X,P}(\Theta) \;=\; \int_X \hat{s}^* \alpha(\hat{\Theta}) \quad (\mathrm{mod}\ 1) \;=\; \int_X (\bar{\phi}^{-1})^* \, \hat{s}'^* \, \hat{\phi}^* \, \alpha(\hat{\Theta}) \quad (\mathrm{mod}\ 1)$$

$$= \int_{X'} \hat{s}'^* \alpha(\hat{\phi}^* \hat{\Theta}) \quad (\mathrm{mod}\ 1) \;=\; S_{X',P'}(\phi^* \Theta)\,.$$

The last equality follows from the definition (3.2) and the fact that $\hat{\phi}^* \hat{\Theta}$ is the $SU(2)$-connection on $\hat{P}'$ induced from the $\mathbb{T}$-connection $\phi^* \Theta$ on P', as can be seen from the relation

$$\begin{aligned} \hat{\phi}^* \hat{\Theta}_{(p,a)} \;&=\; Ad_{a^{-1}} \, \rho_* (\hat{\phi}^* \, pr_1^* \, \Theta_p) + \hat{\phi}^* \, pr_2^* \, \hat{\vartheta}_a \\ &=\; Ad_{a^{-1}} \, \rho_* (pr_1^* \, \phi^* \Theta_p) + pr_2^* \, \hat{\vartheta}_a \,. \end{aligned}$$

(b) is a direct consequence of the definition of integration of differential forms on oriented manifolds.

(c) Let $\hat{\Theta} = \hat{\Theta}_1 \sqcup \hat{\Theta}_2$ be the extension of the $\mathbb{T}$-connection $\Theta = \Theta_1 \sqcup \Theta_2$ on $P = P_1 \sqcup P_2$ to a $SU(2)$-connection on the induced $SU(2)$-bundle $\hat{P} = \hat{P}_1 \sqcup \hat{P}_2$, where $\hat{P}_i = P_i \times_{\mathbb{T}} SU(2)$. Take sections $\hat{s}_i : X_i \to \hat{P}_i$ and let $\hat{s} = \hat{s}_1 \sqcup \hat{s}_2 : X \to \hat{P}$. Then we have

$$S_{X,P}(\Theta) \;=\; \int_X \hat{s}^* \alpha(\hat{\Theta}) \quad (\mathrm{mod}\ 1) \;=\; \int_{X_1} \hat{s}_1^* \alpha(\hat{\Theta}_1) + \int_{X_2} \hat{s}_2^* \alpha(\hat{\Theta}_2) \quad (\mathrm{mod}\ 1)$$

$$= S_{X_1,P_1}(\Theta_1) + S_{X_2,P_2}(\Theta_2)$$

$\square$

As a particular case of (3.6(a)) we have:

Proposition 3.7. *The functional $S_{X,P} : \mathcal{A}_P \to \mathbb{R}/\mathbb{Z}$ for a closed 3-manifold X is invariant under the group of gauge transformations, that is,*

$$S_{X,P}(\Theta \cdot u) \;=\; S_{X,P}(\Theta)\,,$$

for any $u \in \mathcal{G}_X$. Hence $S_{X,P}$ descends to a functional on the quotient space $\mathcal{A}_P/\mathcal{G}_P$ of gauge equivalence classes of connections on P.

We also have the property:

Proposition 3.8. *The stationary points of the functional $S_{X,P} : \mathcal{A}_P \to \mathbb{R}/\mathbb{Z}$ are the flat connections, that is, $d\, S_{X,P}(\Theta) = 0$ if and only if $F_\Theta = d\Theta = 0$.*

Proof. Since $\mathcal{A}_P$ is an affine space, it suffices to consider the variation of $S_{X,P}$ along lines $\Theta_t = \Theta + t\omega,\ t \in \mathbb{R}$, with $\omega \in 2\pi \mathrm{i}\Omega^1(X;\mathbb{R})$. According to Lemma (3.18) and since $\partial X = \emptyset$, we have

$$S_{X,P}(\Theta_t) = S_{X,P}(\Theta) + 2t \int\limits_X \langle F_\Theta \wedge \omega\rangle + t^2 \int\limits_X \langle \omega \wedge d\omega\rangle \quad (\mathrm{mod}\ 1).$$

Thus $\frac{d}{dt}\Big|_{t=0} S_{X,P}(\Theta_t) = 2 \int\limits_X \langle F_\Theta \wedge \omega\rangle$ which proves the proposition. $\square$

3.3. **The Chern-Simons functional for a 3-manifold with boundary.** Let X be a compact oriented 3-manifold with $\partial X \neq \emptyset$. Let $P \to X$ be a $\mathbb{T}$-bundle with first Chern class torsion. Then the $\mathbb{T}$-bundle $\partial P \to \partial X$ obtained by restricting P to the boundary ∂X is trivializable. As before, we consider the induced $SU(2)$-bundle $\hat{P} = P \times_{\mathbb{T}} SU(2)$ on X and the bundle morphism $\rho_P : P \to \hat{P}$. For any section $s : \partial X \to \partial P$, we obtain a section $\rho_P \circ s : \partial X \to \partial \hat{P}$ of the restriction of the $SU(2)$-bundle $\hat{P}$ to ∂X. Since the group $SU(2)$ has $\pi_0(SU(2)) = \pi_1(SU(2)) = \pi_2(SU(2)) = 0$, we can extend the section $\rho_P \circ s$ over ∂X to a global section $\hat{s} : X \to \hat{P}$. We make the following definition.

Definition 3.9. The Chern-Simons functional of a $\mathbb{T}$-connection Θ on $P \to X$ and a section $s : \partial X \to \partial P$ is defined by

$$S_{X,P}(s, \Theta) \ = \int\limits_X \hat{s}^* \alpha(\hat{\Theta}) \quad (\mathrm{mod}\ 1)\,,$$

where $\alpha(\hat{\Theta})$ is the Chern-Simons form of the induced $SU(2)$-connection $\hat{\Theta}$ on the bundle $\hat{P} = P \times_{\mathbb{T}} SU(2)$.

We have to check that the definition of $S_{X,P}(s, \Theta)$ does not depend on the choice of extension $\hat{s}$ of the section $\rho_P \circ s : \partial X \to \partial \hat{P}$. So let $\hat{s}_1 : X \to \hat{P}$ be another extension. Then there is a map $a : X \to SU(2)$ such that $\hat{s}_1(x) = \hat{s}(x) \cdot a(x)$ and

$a(x) = e$, for all $x \in \partial X$, where e denotes the identity element in $SU(2)$. From (3.3) we get

$$\int_X \hat{s}_1^* \alpha(\hat{\Theta}) - \int_X \hat{s}^* \alpha(\hat{\Theta}) \;=\; \int_{\partial X} \langle Ad_{a^{-1}} \hat{s}^* \hat{\Theta} \wedge a^* \hat{\vartheta} \rangle^\flat - \int_X \frac{1}{6}\, a^* \langle \hat{\vartheta} \wedge [\hat{\vartheta}, \hat{\vartheta}] \rangle^\flat \tag{3.10}$$

That $S_{X,P}(s, \Theta)$ is well-defined by (3.9) follows now from the fact that, since $a\big|_{\partial X} = e$, the first integral on the r.h.s of (3.10) vanishes and the second term is an integer. To justify the last assertion we note the following:

Remark 3.11. For each $a : X \to SU(2)$, let $W(a)$ denote the functional

$$W_{\partial X}(a) \;=\; -\int_X \frac{1}{6}\, a^* \langle \hat{\vartheta} \wedge [\hat{\vartheta}, \hat{\vartheta}] \rangle^\flat$$

Then, following the argument in ([F], §2), we let X' be a compact oriented 3-manifold with $\partial X' = \partial X$ and $a' : X' \to SU(2)$ a map such that $a'\big|_{\partial X'} = a\big|_{\partial X}$. Let $\tilde{X} = X \cup (-X')$ be the closed 3-manifold obtained by gluing X and X' along their common boundary. The maps a and a' patch together into a map $\tilde{a} : \tilde{X} \to SU(2)$ and we have

$$W_{\partial X}(a) - W_{\partial X'}(a') \;=\; -\int_{\tilde{X}} \frac{1}{6}\, \tilde{a}^* \langle \hat{\vartheta} \wedge [\hat{\vartheta}, \hat{\vartheta}] \rangle^\flat \;\in \mathbb{Z} \tag{3.12}$$

Since $\tilde{X}$ is closed the r.h.s. is an integer due to the normalization of $\langle \cdot, \cdot \rangle^\flat$. Thus $W_{\partial X}(a)$ depends modulo integers only on the restriction of $a : X \to SU(2)$ to ∂X.

If $a\big|_{\partial X} = e$, we can choose $a' : X' \to SU(2)$ with $a'\big|_{\partial X'} = a\big|_{\partial X}$ to be the constant map $a'(x') = e$ for all $x' \in X'$. Then $W_{\partial X'}(a') = 0$ and therefore $W_{\partial X}(a) \in \mathbb{Z}$.

Remark 3.13. As in (3.5), if $P \to X$ is a trivializable $\mathbb{T}$-bundle and $s : X \to P$ a global section, then for any connection Θ on P the Chern-Simons functional $S_{X,P}(s, \Theta)$ is given by the expression

$$S_{X,P}(s, \Theta) \;=\; \int_X s^* \langle \Theta \wedge F_\Theta \rangle \pmod 1 \tag{3.14}$$

The following proposition describes the dependence of the Chern-Simons functional $S_{X,P}(s, \Theta)$ on the section $s : \partial X \to \partial P$.

Proposition 3.15. *Let $s, s_1 : \partial X \to \partial P$ be two sections of the restriction of the $\mathbb{T}$-bundle $P \to X$ to ∂X and let Θ be a connection on P. Then*

$$S_{X,P}(s_1, \Theta) \;=\; S_{X,P}(s, \Theta) + \int\limits_{\partial X} \langle s^*\Theta \wedge \lambda^*\vartheta \rangle \quad (\mathrm{mod}\ 1)\,,$$

where $\lambda : \partial X \to \mathbb{T}$ is the map defined by $s_1(x) = s(x) \cdot \lambda(x)$, for all $x \in \partial X$.

Proof. The sections $\rho_P \circ s$, $\rho_P \circ s_1 : \partial X \to \partial \hat{P}$ extend to sections $\hat{s}$, $\hat{s}_1 : X \to \hat{P}$ of the induced $SU(2)$-bundle $\hat{P} = P \times_{\mathbb{T}} SU(2)$. The map $a : X \to SU(2)$ defined by $\hat{s}_1(x) = \hat{s}(x) \cdot a(x)$, for any $x \in X$, restricts to $a\big|_{\partial X} = \rho \circ \lambda$ over the boundary of X. The difference $S_{X,P}(s_1, \Theta) - S_{X,P}(s, \Theta)$ is given by an expression identical to the r.h.s. of the equation (3.10). The second term $W_{\partial X}(a)$ is also in this case an integer. To see that choose in (3.12) a 3-manifold X' with $\partial X' = \partial X$, for which the map $\lambda : \partial X \to \mathbb{T}$ extends to a map $\lambda' : X' \to \mathbb{T}$. Let $a' = \rho \circ \lambda' : X' \to SU(2)$. Then, since $\rho^* \hat{\vartheta} = \rho_* \vartheta$, we get that

$$W_{\partial X'}(a') = -\int\limits_{X'} \frac{1}{6}\, a'^* \langle \hat{\vartheta} \wedge [\hat{\vartheta}, \hat{\vartheta}] \rangle^\flat = -\int\limits_{X'} \frac{1}{6}\, \lambda'^* \langle \vartheta \wedge [\vartheta, \vartheta] \rangle = 0$$

Thus (3.12) implies that $W_{\partial X}(a) \in \mathbb{Z}$. The first term on the r.h.s. of (3.10) gives

$$\begin{aligned} \int\limits_{\partial X} \langle Ad_{a^{-1}} \hat{s}^* \hat{\Theta} \wedge a^* \hat{\vartheta} \rangle^\flat \;&=\; \int\limits_{\partial X} \langle Ad_{(\rho \circ \lambda)^{-1}}\, s^* \rho_P^* \hat{\Theta} \wedge \lambda^* \rho^* \hat{\vartheta} \rangle^\flat \\ &= \int\limits_{\partial X} \langle \rho_* \left(Ad_{\lambda^{-1}} s^* \Theta \right) \wedge \rho_* \left(\lambda^* \vartheta \right) \rangle^\flat \\ &= \int\limits_{\partial X} \langle s^* \Theta \wedge \lambda^* \vartheta \rangle \end{aligned}$$

which proves the proposition. ∎

Similarly to Theorem (3.6) we have

Theorem 3.16. *The Chern-Simons functional $S_{X,P}(\cdot, \cdot)$, defined for a compact oriented 3-manifold X with nonempty boundary and a principal $\mathbb{T}$-bundle $P \to X$ with first Chern class torsion, has the following properties:*

(a) Functoriality

Let $\phi : P' \to P$ be a morphism of principal $\mathbb{T}$-bundles covering an orientation

preserving diffeomorphism $\bar{\phi} : X' \to X$. *If* Θ *is a connection on* P *and* $s' : \partial X' \to \partial P'$ *a section of* P' *over the boundary of* X'*, then*

$$S_{X',P'}(s', \phi^*\Theta) \;=\; S_{X,P}(\partial\phi \circ s' \circ \partial\bar{\phi}^{-1}, \Theta)\,,$$

where $\partial\phi : \partial P' \to \partial P$ *and* $\partial\bar{\phi} : \partial X' \to \partial X$ *are restrictions to the boundary.*

(b) Orientation

If $-X$ *denotes the manifold* X *with the opposite orientation and* $s : \partial X \to \partial P$*, then*

$$S_{-X,P}(s, \Theta) \;=\; -S_{X,P}(s, \Theta)$$

(c) Disjoint union

Let X *be the disjoint union* $X = X_1 \sqcup X_2$ *and* $P = P_1 \sqcup P_2$ *a principal* $\mathbb{T}$*-bundle over* X *with first Chern class torsion. If* Θ_i *are connections on* $P_i \to X_i$ *and* $s_i : \partial X_i \to \partial P_i$ *sections over the boundary, then*

$$S_{X_1 \sqcup X_2, P_1 \sqcup P_2}(s_1 \sqcup s_2, \Theta_1 \sqcup \Theta_2) \;=\; S_{X_1,P_1}(s_1, \Theta_1) \,+\, S_{X_2,P_2}(s_2, \Theta_2)$$

Proof. (a) Let $\hat{P} = P \times_{\mathbb{T}} SU(2)$ and $\hat{P}' = P' \times_{\mathbb{T}} SU(2)$ be the induced $SU(2)$-bundles, $\hat{\phi} : \hat{P}' \to \hat{P}$ the induced bundle map and $\hat{\Theta}$ the induced connection on $\hat{P}$. The sections $s' : \partial X' \to \partial P'$ and $s = \partial\phi \circ s' \circ \partial\bar{\phi}^{-1} : \partial X \to \partial P$ determine sections $\rho_{P'} \circ s' : \partial X' \to \partial \hat{P}'$ and $\rho_P \circ s : \partial X \to \partial \hat{P}$. Let $\hat{s}' : X' \to \hat{P}'$ be an extension of $\rho_{P'} \circ s'$. Then, using the fact that $\hat{\phi} \circ \rho_{P'} = \rho_P \circ \phi$, we can show that $\hat{s} = \hat{\phi} \circ \hat{s}' \circ \bar{\phi}^{-1} : X \to \hat{P}$ is an extension of $\rho_P \circ s$. Moreover, the $SU(2)$-connection on $\hat{P}'$ induced from the $\mathbb{T}$-connection $\phi^*\Theta$ on P' is $\hat{\phi}^*\hat{\Theta}$. Applying the definition (3.9) we get

$$\begin{aligned} S_{X,P}(s, \Theta) \;&=\; \int\limits_X \hat{s}^*\alpha(\hat{\Theta}) \pmod 1 \;=\; \int\limits_X (\bar{\phi}^{-1})^* \hat{s}'^* \hat{\phi}^* \alpha(\hat{\Theta}) \pmod 1 \\ &=\; \int\limits_{X'} \hat{s}'^* \alpha(\hat{\phi}^*\hat{\Theta}) \pmod 1 \;=\; S_{X',P'}(s', \phi^*\Theta) \end{aligned}$$

(b) and (c) are obvious. $\square$

As a consequence of (3.16) and (3.15) we have:

Proposition 3.17. *The functional $S_{X,P}(\cdot,\cdot)$ changes under the group of gauge transformations as*

$$S_{X,P}(s,\Theta\cdot u) \;=\; S_{X,P}(s,\Theta) + \int\limits_{\partial X} \langle s^*\Theta \wedge u^*\vartheta\rangle \quad (\mathrm{mod}\ 1)\,,$$

for any $u \in \mathcal{G}_X = \mathrm{Map}(X,\mathbb{T})$.

Proof. Let $\phi_u : P \to P$ denote the bundle automorphism determined by u. Then

$$\begin{aligned} S_{X,P}(s,\Theta\cdot u) \;&=\; S_{X,P}(s,\phi_u^*\Theta) \;=\; S_{X,P}(\partial\phi_u \circ s,\Theta) && (\text{by } (3.16)(a)) \\ &=\; S_{X,P}(s,\Theta) + \int\limits_{\partial X} \langle s^*\Theta \wedge u^*\vartheta\rangle \quad (\mathrm{mod}\ 1) && (\text{by } (3.15)) \end{aligned}$$

□

We also note the following lemma:

Lemma 3.18. *Let X be a compact oriented 3-manifold and $P \to X$ a $\mathbb{T}$-bundle with first Chern class torsion. If Θ and Θ' are connections on P and $s : \partial X \to \partial P$ a section over the boundary of X, then*

$$\begin{aligned} S_{X,P}(s,\Theta') - S_{X,P}(s,\Theta) \;=\; &- \int\limits_{\partial X} s^*\langle\Theta\wedge\omega\rangle \\ &+ 2\int\limits_X \langle F_\Theta \wedge \omega\rangle + \int\limits_X \langle\omega\wedge d\omega\rangle \quad (\mathrm{mod}\ 1)\,, \end{aligned}$$

where $\omega = \Theta' - \Theta$.

Proof. Let $\hat\Theta$ and $\hat\Theta'$ be the $SU(2)$-connections on $\hat P = P \times_{\mathbb{T}} SU(2)$ induced from the $\mathbb{T}$-connections Θ and Θ' on P. It follows from (3.1) that the $\mathfrak{su}(2)$-valued 1-form $\hat\omega = \hat\Theta' - \hat\Theta$ on $\hat P$ is related to ω by

$$\hat\omega_{[p,a]} \;=\; Ad_{a^{-1}}(\rho_*\, pr_1^*\, \omega_p)\,. \tag{3.19}$$

and $\rho_P^*\hat\omega \;=\; \rho_P^*(\hat\Theta' - \hat\Theta) \;=\; \rho_*(\Theta' - \Theta) \;=\; \rho_*\omega$. Routine algebraic computations give us for the difference of the Chern-Simons forms of the $SU(2)$-connections $\hat\Theta'$

and $\hat{\Theta}$ the expression:

$$\alpha(\hat{\Theta}') - \alpha(\hat{\Theta}) \;=\; -\, d\langle \hat{\Theta} \wedge \hat{\omega} \rangle^\flat + 2\langle F_{\hat{\Theta}} \wedge \hat{\omega} \rangle^\flat \\ + \langle \hat{\omega} \wedge (d\hat{\omega} + [\hat{\Theta}, \hat{\omega}]) \rangle^\flat + \frac{1}{3}\langle \hat{\omega} \wedge [\hat{\omega}, \hat{\omega}] \rangle^\flat$$

According to (3.19) the $\mathfrak{su}(2)$-valued 1-form $\hat{\omega}$ comes from a Lie $\mathbb{T}$-valued form ω, so we have $[\hat{\omega}, \hat{\omega}] = 0$. Thus the above expression reduces to

$$\alpha(\hat{\Theta}') - \alpha(\hat{\Theta}) \;=\; -d\langle \hat{\Theta} \wedge \hat{\omega} \rangle^\flat + 2\langle F_{\hat{\Theta}} \wedge \hat{\omega} \rangle^\flat + \langle \hat{\omega} \wedge d\hat{\omega} \rangle^\flat$$

The 3-forms $\langle F_{\hat{\Theta}} \wedge \hat{\omega} \rangle^\flat$ on $\hat{P}$ and $\langle F_\Theta \wedge \omega \rangle$ on P are both lifts to $\hat{P}$ and P, respectively, of the same 3-form on X. The same statement applies to the 3-forms $\langle \hat{\omega} \wedge d\hat{\omega} \rangle^\flat$ on $\hat{P}$ and $\langle \omega \wedge d\omega \rangle$ on P. Letting $\hat{s} : X \to \hat{P}$ be the extension to X of the section $\rho_P \circ s : \partial X \to \partial \hat{P}$, we can write

$$\int_X \hat{s}^* d\langle \hat{\Theta} \wedge \hat{\omega} \rangle^\flat = \int_{\partial X} s^* \rho_P^* \langle \hat{\Theta} \wedge \hat{\omega} \rangle^\flat = \int_{\partial X} s^* \langle \rho_* \Theta \wedge \rho_* \omega \rangle^\flat = \int_{\partial X} s^* \langle \Theta \wedge \omega \rangle$$

Putting all these results together, we find that the difference

$$S_{X,P}(s, \Theta') - S_{X,P}(s, \Theta) = \int_X \hat{s}^* \alpha(\hat{\Theta}') - \int_X \hat{s}^* \alpha(\hat{\Theta})$$

is given by the expression in (3.18). □

Lemma 3.20. *Let Θ be a connection on the $\mathbb{T}$-bundle $P \to X$ with $c_1(P)$ torsion and $s : \partial X \to \partial P$ a section over the boundary of X. If $u : X \to \mathbb{T}$ is an element of $\mathcal{G}_X$, then*

$$\int_X \langle F_\Theta \wedge u^* \vartheta \rangle \;=\; \int_{\partial X} \langle s^* \Theta \wedge u^* \vartheta \rangle \qquad (\text{mod } 1)$$

Proof. The fact that the integral $\int_X \langle F_\Theta \wedge u^* \vartheta \rangle$ depends modulo $\mathbb{Z}$ only on the boundary data can be seen as follows. Let X' be a compact oriented 3-manifold with $\partial X' = \partial X$ and such that the map $u : \partial X \to \mathbb{T}$ extends to a map $u' : X' \to \mathbb{T}$. Then glue the manifolds X and X' along their common boundary into the closed 3-manifold $\tilde{X} = X \sqcup (-X')$ and extend the connection $\partial\Theta$ on ∂X to a connection

Θ' over X'. Let $\tilde{\Theta} = \Theta \sqcup \Theta'$ and let $\tilde{u} : \tilde{X} \to \mathbb{T}$ denote the map with restrictions $\tilde{u}\big|_X = u$ and $\tilde{u}\big|_{X'} = u'$. The integral

$$\int\limits_{\tilde{X}} \langle F_{\tilde{\Theta}} \wedge \tilde{u}^* \vartheta \rangle \;=\; \int\limits_X \langle F_\Theta \wedge u^* \vartheta \rangle \;-\; \int\limits_{X'} \langle F_{\Theta'} \wedge u'^* \vartheta \rangle \;\in \mathbb{Z}\,, \tag{3.21}$$

since $\frac{F_{\tilde{\Theta}}}{2\pi \mathrm{i}}$ and $\frac{\tilde{u}^* \vartheta}{2\pi \mathrm{i}}$ represent integral cohomology classes. Thus $\int\limits_X \langle F_\Theta \wedge u^* \vartheta \rangle$ depends modulo $\mathbb{Z}$ only on the restrictions $\partial\Theta$ and $u\big|_{\partial X}$ to the boundary ∂X. To prove the lemma take Θ' extending $\partial\Theta$ over X' to be a connection on the trivial bundle $\pi' : P' = X' \times \mathbb{T} \to X'$. Then if $s' : X' \to P'$ is a section we have from (3.21)

$$\begin{aligned}
\int\limits_X \langle F_\Theta \wedge u^* \vartheta \rangle \;&=\; \int\limits_{X'} \langle F_{\Theta'} \wedge u'^* \vartheta \rangle \pmod 1 = \int\limits_{X'} s'^* \langle d\Theta' \wedge \pi'^* u'^* \vartheta \rangle \pmod 1 \\
&= \int\limits_{X'} d\, s'^* \langle \Theta' \wedge \pi'^* u'^* \vartheta \rangle \pmod 1 = \int\limits_{\partial X'} \langle s'^* \Theta' \wedge u'^* \vartheta \rangle \pmod 1 \\
&= \int\limits_{\partial X} \langle s^* \Theta \wedge u^* \vartheta \rangle \pmod 1\,,
\end{aligned}$$

where $s : \partial X \to \partial P$ is any section over the boundary. $\square$

Proposition 3.22. *Let $P \to X$ be a $\mathbb{T}$-bundle with first Chern class torsion. For any section $s : \partial X \to \partial P$ the functional $S_{X,P}(s, \cdot) : \mathcal{A}_P^f \to \mathbb{R}/\mathbb{Z}$ is invariant under the group $\mathcal{G}_X$ of gauge transformations, that is,*

$$S_{X,P}(s, \Theta \cdot u) \;=\; S_{X,P}(s, \Theta)\,,$$

for any $u \in \mathcal{G}_X$. Hence $S_{X,P}(s, \cdot)$ descends to a functional on the moduli space $\mathcal{M}_P$ of flat connections on P.

Proof. If Θ is flat, then Lemma (3.20) gives

$$\int\limits_{\partial X} \langle s^* \Theta \wedge u^* \vartheta \rangle \;=\; 0 \pmod 1\,,$$

for any $u \in \mathcal{G}_X$. The proposition follows now from Prop. (3.17) together with the above result. $\square$

4. The prequantum line bundle

4.1. **The line bundle over the moduli space of flat connections.** As shown in Sect.2 to each closed oriented 2-dimensional manifold Σ there corresponds the symplectic torus $(\mathcal{M}_\Sigma, \omega_\Sigma)$ of gauge equivalence classes of flat $\mathbb{T}$-connections on Σ.

Given a positive even integer k, we construct in this section a hermitian line bundle $\mathcal{L}_\Sigma$ over $\mathcal{M}_\Sigma$ with a unitary connection with curvature $-2\pi\mathrm{i}k\omega_\Sigma$. In the language of geometric quantization such a line bundle is called a *prequantum line bundle* for the symplectic manifold $(\mathcal{M}_\Sigma, k\omega_\Sigma)$. The set of prequantum line bundles for $(\mathcal{M}_\Sigma, k\omega_\Sigma)$ is a principal homogeneous space for the cohomology group $H^1(\mathcal{M}_\Sigma; \mathbb{T})$. Since $H^1(\mathcal{M}_\Sigma; \mathbb{T})$ is nontrivial, by general geometric quantization arguments the choice of the prequantum line bundle is not unique. However, the 'Chern-Simons line construction' from ([F], §2), which we are going to apply here, singles out a prequantum line bundle $\mathcal{L}_\Sigma$.

Let $Q \to \Sigma$ be a trivializable $\mathbb{T}$-bundle over the closed oriented surface Σ. To each connection η on Q we are going to associate a hermitian line L_η by the following construction. Let $\Gamma(\Sigma; Q)$ stand for the space of sections of the line bundle $Q \to \Sigma$. Any two sections $s, s_1 \in \Gamma(\Sigma; Q)$ are related by an element $\lambda : \Sigma \to \mathbb{T}$ of the group of gauge transformations $\mathcal{G}_\Sigma$, defined by $s_1(x) = s(x) \cdot \lambda(x)$. Thus we let L_η be the space of functions $f : \Gamma(\Sigma; Q) \to \mathbb{C}$ satisfying the relation

$$f(s_1) \, = \, f(s) \, c_\Sigma(s^*\eta, \lambda) \, , \tag{4.1}$$

where $c_\Sigma(s^*\eta, \lambda)$ is the $\mathbb{T}$-valued cocycle

$$\begin{aligned} c_\Sigma \ &: \ 2\pi\mathrm{i} \, \Omega^1(\Sigma; \mathbb{R}) \times \mathcal{G}_\Sigma \longrightarrow \mathbb{T} \\ c_\Sigma(\alpha, \lambda) \, &= \, \mathrm{e}^{\pi\mathrm{i}k \int\limits_\Sigma \langle \alpha \wedge \lambda^* \vartheta \rangle} \, = \, \mathrm{e}^{-\pi\mathrm{i}k \int\limits_\Sigma \frac{\alpha}{2\pi\mathrm{i}} \wedge \lambda^*(\frac{\vartheta}{2\pi\mathrm{i}})} \, . \end{aligned} \tag{4.2}$$

c_Σ satisfies the cocycle identity

$$c_\Sigma(\alpha, \lambda_1\lambda_2) \, = \, c_\Sigma(\alpha, \lambda_1) \, c_\Sigma(\alpha + \lambda_1^*\vartheta, \lambda_2) \tag{4.3}$$

if and only if

$$\frac{k}{2}\int_{\Sigma} \lambda_1^*(\frac{\vartheta}{2\pi \mathrm{i}}) \wedge \lambda_2^*(\frac{\vartheta}{2\pi \mathrm{i}}) \in \mathbb{Z}\,,$$

for any $\lambda_1, \lambda_2 \in \mathcal{G}_\Sigma$. This imposes the restriction $k \in 2\mathbb{Z}$. Hence, for the remainder of this paper, we are going to assume that k is a positive even integer.

An element $f \in L_\eta$ is uniquely determined by its value at a single point $s \in \Gamma(\Sigma; Q)$. Therefore $\dim_{\mathbb{C}} L_\eta = 1$. We also note that a section $s : \Sigma \to Q$ determines a trivialization $L_\eta \cong \mathbb{C}$ by mapping $f \mapsto f(s)$. The complex line L_η has a hermitian inner product determined by the standard inner product of $\mathbb{C}$: $(f_1, f_2) = \overline{f_1(s)} f_2(s)$.

As in [F] we have the property that, as η varies over the space $\mathcal{A}_Q$ of connections on Q, the lines L_η fit together smoothly into a hermitian line bundle $\hat{\mathcal{L}}_Q \to \mathcal{A}_Q$. Moreover, we have:

Theorem 4.4. *Let Σ be a closed oriented 2-manifold. The assignment*

$$\eta \mapsto L_{\Sigma,\eta} = L_\eta\,,$$

for η connections on trivializable $\mathbb{T}$-bundles over Σ, is smooth and satisfies:
(a) Functoriality
Let $\psi : Q' \to Q$ be a morphism between trivializable $\mathbb{T}$-bundles covering an orientation preserving diffeomorphism $\bar{\psi} : \Sigma' \to \Sigma$ and let η be a connection on Q. Then there is an induced isometry

$$\begin{aligned} \psi^* \ : L_\eta &\longrightarrow L_{\psi^*\eta} \\ f \ &\longmapsto \psi^* f\,, \end{aligned}$$

where $\psi^* f \in L_{\psi^*\eta}$ is defined by $(\psi^* f)(s') = f(\psi \circ s' \circ \bar{\psi}^{-1})$, for any $s' \in \Gamma(\Sigma'; Q')$.
(b) Orientation
If $-\Sigma$ denotes the manifold Σ with the opposite orientation, then there is a natural isometry

$$L_{-\Sigma,\eta} \cong \overline{L_{\Sigma,\eta}}$$

(c) Disjoint union

If $\Sigma = \Sigma_1 \sqcup \Sigma_2$ is a disjoint union and η_i are connections on trivializable $\mathbb{T}$-bundles over Σ_i, then there is a natural isometry

$$L_{\eta_1 \sqcup \eta_2} \;\cong\; L_{\eta_1} \otimes L_{\eta_2}$$

Proof. (a) We have to check that $\psi^* f$ does indeed belong to the line $L_{\psi^*\eta}$. So let $\lambda' : \Sigma' \to \mathbb{T}$ and let $s', s'_1 \in \Gamma(\Sigma'; Q')$ be sections related by $s'_1(x') = s'(x') \cdot \lambda'(x')$, for all $x' \in \Sigma'$. Then $(\psi \circ s'_1 \circ \bar{\psi}^{-1})(x) = (\psi \circ s' \circ \bar{\psi}^{-1})(x) \cdot \lambda(x)$ at any $x \in \Sigma$, where $\lambda = \lambda' \circ \bar{\psi}^{-1} : \Sigma \to \mathbb{T}$. Therefore we obtain

$$\begin{aligned}
(\psi^* f)(s'_1) &= f(\psi \circ s'_1 \circ \bar{\psi}^{-1}) \\
&= f(\psi \circ s' \circ \bar{\psi}^{-1})\, c_\Sigma((\bar{\psi}^{-1})^* s'^* \psi^* \eta, \lambda) \qquad \text{(by definition (4.1))} \\
&= (\psi^* f)(s')\, c_{\Sigma'}(s'^*(\psi^*\eta), \lambda') \qquad (\bar{\psi} \text{ is orientation preserving})
\end{aligned}$$

(b) follows from the fact that the $\mathbb{T}$-valued cocycle c_Σ changes to the complex conjugate expression when the orientation of Σ is reversed.

(c) η_i are connections on trivializable $\mathbb{T}$-bundles $Q_i \to \Sigma_i$. Let $s_i : \Sigma_i \to Q_i$ be some sections and $\lambda_i : \Sigma_i \to \mathbb{T}$ gauge transformations and set $s = s_1 \sqcup s_2$, $\eta = \eta_1 \sqcup \eta_2$ and $\lambda = \lambda_1 \sqcup \lambda_2$. Then $c_\Sigma(s^*\eta, \lambda) = c_{\Sigma_1}(s_1^*\eta_1, \lambda_1)\, c_{\Sigma_2}(s_2^*\eta_2, \lambda_2)$ and this induces the isometry

$$\begin{aligned}
L_{\eta_1} \otimes L_{\eta_2} &\longrightarrow L_{\eta_1 \sqcup \eta_2} \\
f_1 \otimes f_2 &\longmapsto f\,,
\end{aligned}$$

where $f(s) = f_1(s_1) f_2(s_2)$, for any section $s = s_1 \sqcup s_2$. □

Each section $s : \Sigma \to Q$ induces a trivialization of the line bundle $\hat{\mathcal{L}}_Q$, given by the unitary section

$$\gamma_s : \mathcal{A}_Q \longrightarrow \hat{\mathcal{L}}_Q$$

with $\gamma_s(\eta) \in L_\eta$ defined, for each η, by $(\gamma_s(\eta))(s) = 1$. We define the connection $\hat{\nabla}$ on $\hat{\mathcal{L}}_Q$ by

$$\hat{\nabla} \gamma_s \;=\; -2\pi \mathrm{i} k\, \hat{\theta}_s\, \gamma_s\,, \tag{4.5}$$

where $\hat\theta_s$ is the 1-form on $\mathcal{A}_Q$ given by

$$(\dot\eta \lrcorner \hat\theta_s)_\eta \,=\, -\frac{1}{2}\int\limits_\Sigma s^*\langle \eta\wedge\dot\eta\rangle\,, \tag{4.6}$$

for any $\eta\in\mathcal{A}_Q$ and $\dot\eta\in T_\eta\mathcal{A}_Q$.

Let $\mathbb{C}^\times = \mathbb{C}\smallsetminus\{0\}$ and let $\hat{\mathcal{L}}_Q^\times \to \mathcal{A}_Q$ be the principal $\mathbb{C}^\times$-bundle associated to $\pi : \hat{\mathcal{L}}_Q \to \mathcal{A}_Q$ (the bundle $\hat{\mathcal{L}}_Q^\times$ is obtained from $\hat{\mathcal{L}}_Q$ by removing the zero section). The connection form $-2\pi\mathrm{i}k\,\hat\alpha$ on $\hat{\mathcal{L}}_Q^\times$ corresponding to $\hat\nabla$ is defined by the expression

$$\hat\alpha \,=\, \pi^*\hat\theta_s - \frac{1}{2\pi\mathrm{i}k}\,\Phi_s^*\,pr_2^*\vartheta\,, \tag{4.7}$$

where $\Phi_s : \hat{\mathcal{L}}_Q \to \mathcal{A}_Q\times\mathbb{C}$ is the trivializing map defined by $\Phi_s(\gamma_s(\eta)) = (\eta,1)$ and $pr_2 : \mathcal{A}_Q\times\mathbb{C}\to\mathbb{C}$ the natural projection. Obviously we have $\gamma_s^*\hat\alpha = \hat\theta_s$. If $\psi\in\mathcal{G}_Q = \mathrm{Aut}(Q)$, the new section $\psi\circ s : \Sigma\to Q$ induces a new section $\gamma_{\psi\circ s} : \mathcal{A}_Q\to\hat{\mathcal{L}}_Q$ related to γ_s by

$$\gamma_{\psi\circ s}(\eta) \,=\, c_\Sigma(s^*\eta,\lambda_\psi)^{-1}\,\gamma_s(\eta)\,, \tag{4.8}$$

where $\lambda_\psi : \Sigma\to\mathbb{T}$ is the map associated to ψ. The connection form $\hat\theta_{\psi\circ s}$ is related to $\hat\theta_s$ by

$$\begin{aligned}(\dot\eta \lrcorner \hat\theta_{\psi\circ s})_\eta \,&=\, -\frac{1}{2}\int\limits_\Sigma s^*\psi^*\langle\eta\wedge\dot\eta\rangle \,=\, -\frac{1}{2}\int\limits_\Sigma \langle (s^*\eta+\lambda_\psi^*\vartheta)\wedge s^*\dot\eta\rangle\\ &=\, (\dot\eta \lrcorner \hat\theta_s)_\eta + \frac{1}{2}\int\limits_\Sigma\langle s^*\dot\eta\wedge\lambda_\psi^*\vartheta\rangle\end{aligned}$$

Taking the differential of the cocycle $c_\Sigma(s^*\eta,\lambda_\psi)$ with respect to η and using the above expression, we find that

$$(\hat\theta_{\psi\circ s})_\eta \,=\, (\hat\theta_s)_\eta + \frac{1}{2\pi\mathrm{i}k}\,\frac{d\,c_\Sigma(s^*\eta,\lambda_\psi)}{c_\Sigma(s^*\eta,\lambda_\psi)}$$

Together with (4.8) this implies that

$$\pi^*\hat\theta_{\psi\circ s} - \frac{1}{2\pi\mathrm{i}k}\Phi_{\psi\circ s}^*\,pr_2^*\vartheta \,=\, \pi^*\hat\theta_s - \frac{1}{2\pi\mathrm{i}k}\Phi_s^*\,pr_2^*\vartheta\,,$$

which shows that the definition (4.7) of $\hat\alpha$ does not depend on the choice of trivialization of $\hat{\mathcal{L}}_Q\to\mathcal{A}_Q$.

Since γ_s is a unitary section and $\hat{\theta}_s$ is real, the connection $\hat{\nabla}$ is compatible with the hermitian structure on $\hat{\mathcal{L}}_Q$. The curvature of $\hat{\nabla}$ is $-2\pi \mathrm{i} k\omega_\Sigma$, where

$$\omega_\Sigma(\dot\eta, \dot\eta') \;=\; (d\hat{\theta}_s)(\dot\eta, \dot\eta') \;=\; -\int_\Sigma \langle \dot\eta \wedge \dot\eta' \rangle \tag{4.9}$$

is the standard symplectic form on $T\mathcal{A}_Q \cong 2\pi\mathrm{i}\,\Omega^1(\Sigma;\mathbb{R})$.

It follows from (4.4 (a)) that the action of the group of gauge transformations $\mathcal{G}_Q$ on $\mathcal{A}_Q$ lifts to the line bundle $\hat{\mathcal{L}}_Q$ preserving the hermitian metric. The lift of $\mathcal{G}_Q$ to $\hat{\mathcal{L}}_Q$ preserves also the connection $\hat{\nabla}$. To prove this we consider the induced action on sections of $\hat{\mathcal{L}}_Q \to \mathcal{A}_Q$:

$$(\psi^* \cdot \gamma_s)(\eta) \;=\; \psi^*\gamma_s(\psi^{*-1}\eta)\,.$$

A simple application of (4.4 (a)) and the definition (4.1) gives that

$$\psi^* \cdot \gamma_s \;=\; \gamma_{\psi^{-1}\circ s}\,. \tag{4.10}$$

The action of $\mathcal{G}_Q$ on $\hat{\mathcal{L}}_Q$ preserves $\hat{\nabla}$ if

$$\Big[\hat{\nabla}_{\psi^*\dot\eta}(\psi^* \cdot \gamma_s)\Big] \;=\; \psi^*\Big[\big(\hat{\nabla}_{\dot\eta}\gamma_s\big)(\eta)\Big]\,.$$

Using (4.5), (4.6) and (4.10), a routine check shows that the above equation is indeed satisfied. The $\mathcal{G}_Q$-action on $\mathcal{A}_Q$ preserves the symplectic form ω_Σ. This follows from the fact that $-2\pi\mathrm{i}k\omega_\Sigma$ is the curvature of $\hat{\nabla}$.

Since the action of $\mathcal{G}_Q$ on $\mathcal{A}_Q$ is symplectic and lifts to the line bundle $\pi : \hat{\mathcal{L}}_Q \to \mathcal{A}_Q$ preserving the hermitian metric and the connection $\hat{\nabla}$, there exists a moment map for this action. Following the general construction in [K], we give below the explicit form of this moment map. The $\mathcal{G}_Q$-action associates to each element $\xi : \Sigma \to \mathrm{Lie}\mathbb{T}$ in $\mathrm{Lie}\,\mathcal{G}_Q$ real vector fields X_ξ on $\mathcal{A}_Q$ and V_ξ on $\hat{\mathcal{L}}_Q$, with $\pi_* V_\xi = X_\xi$. The vector field V_ξ preserves the connection form $-2\pi\mathrm{i}k\,\hat\alpha$ on $\hat{\mathcal{L}}_Q^\times$, i.e. the Lie derivative $L_{V_\xi}\hat\alpha = 0$. Together with the fact that V_ξ preserves the hermitian metric, this implies that the function $\hat\mu_\xi$ on $\mathcal{A}_Q$, defined by $\hat\mu_\xi \circ \pi = V_\xi \lrcorner (k\hat\alpha)$, is real-valued and satisfies $X_\xi \lrcorner (k\omega_\Sigma) + d\hat\mu_\xi = 0$. The map $\mu : \mathcal{A}_Q \to (\mathrm{Lie}\,\mathcal{G}_Q)^*$,

$\eta \mapsto \mu(\eta)$ with $(\mu(\eta))(\xi) = \hat{\mu}_\xi(\eta)$, is the *moment map* of the $\mathcal{G}_Q$-action on $\mathcal{A}_Q$. Using (4.7) we find:

$$\begin{aligned}\hat{\mu}_\xi \circ \pi \;&=\; V_\xi \,\lrcorner\, (k\hat{\alpha}) \;=\; V_\xi \,\lrcorner\, \pi^*(k\hat{\theta}_s) - \frac{1}{2\pi \mathrm{i}} V_\xi \,\lrcorner\, \Phi_s^*\, pr_2^*\vartheta \\ &=\; k(X_\xi \,\lrcorner\, \hat{\theta}_s) \circ \pi - \frac{1}{2\pi \mathrm{i}} V_\xi \,\lrcorner\, \Phi_s^*\, pr_2^*\vartheta \,.\end{aligned}$$

A direct computation gives

$$(X_\xi)_\eta \;=\; \frac{d}{dt}\Big|_{t=0} \eta \cdot \mathrm{e}^{t\xi} \;=\; d\xi$$

and

$$V_\xi \,\lrcorner\, (\Phi_s^*\, pr_2^*\vartheta) \;=\; \Big[(pr_{2*}\Phi_{s*}V_\xi) \,\lrcorner\, \vartheta\Big] \circ pr_2 \circ \Phi_s \;=\; -2\pi \mathrm{i} k (d\xi \,\lrcorner\, \hat{\theta}_s) \circ \pi \,.$$

Therefore we obtain for the moment map the expression

$$\hat{\mu}_\xi(\eta) \;=\; 2k(d\xi \,\lrcorner\, \hat{\theta}_s)_\eta = -k \int_\Sigma \langle s^*\eta \wedge d\xi \rangle \;=\; -k \int_\Sigma \langle s^* d\eta \wedge \xi \rangle \,.$$

The preimage $\mu^{-1}(0)$, which is $\mathcal{G}_Q$-invariant, is the space $\mathcal{A}_Q^f$ of flat connections. Hence the space $\mathcal{M}_Q$ of equivalence classes of flat connections is the symplectic quotient $\mathcal{M}_Q = \mathcal{A}_Q /\!/ \mathcal{G}_Q = \mu^{-1}(0)/\mathcal{G}_Q$. Since $\mathcal{G}_Q$ preserves ω_Σ and $\omega_\Sigma(X_\xi, \dot{\eta}) = 0$, for any element ξ in $\mathrm{Lie}\,\mathcal{G}_Q$ and any $\dot{\eta} \in T\mathcal{A}_Q^f$, the symplectic form ω_Σ on $\mathcal{A}_Q$ pushes down to a symplectic form on $\mathcal{M}_Q$, which we continue to denote by ω_Σ.

The group $\mathcal{G}_Q$ does not act freely on $\mathcal{A}_Q$. However, the stabilizer at a point $\eta \in \mathcal{A}_Q$ is the subgroup $Z \subset \mathcal{G}_Q$ of constant gauge transformations $Z \cong \{\lambda : \Sigma \to \mathbb{T} \mid \lambda = \text{constant}\} \cong \mathbb{T}$ and Z acts trivially on L_η. Therefore the line bundle $\hat{\mathcal{L}}_Q \to \mathcal{A}_Q^f = \mu^{-1}(0)$ pushes down to a line bundle $\mathcal{L}_Q \to \mathcal{M}_Q$. We have the identification

$$\Gamma(\mathcal{M}_Q; \mathcal{L}_Q) \;=\; \Gamma(\mathcal{A}_Q^f; \hat{\mathcal{L}}_Q)^{\mathcal{G}_Q} \tag{4.11}$$

of the space of sections of $\mathcal{L}_Q \to \mathcal{M}_Q$ with the space of $\mathcal{G}_Q$-invariant sections of $\hat{\mathcal{L}}_Q \to \mathcal{A}_Q^f$. The hermitian metric and connection on $\hat{\mathcal{L}}_Q$ also push down to $\mathcal{L}_Q$. A $\mathcal{G}_Q$-invariant section γ of $\hat{\mathcal{L}}_Q \to \mathcal{A}_Q^f$ satisfies $\hat{\nabla}_{X_\xi}\gamma = 0$, for all $\xi : \Sigma \to \mathrm{Lie}\mathbb{T}$.

Having in view the identification (4.11), the push-down connection ∇ on the line bundle $\mathcal{L}_Q \to \mathcal{M}_Q$ is defined by

$$\nabla_X \gamma \,=\, \hat{\nabla}_{\hat{X}} \gamma \,,$$

for any $X \in T\mathcal{M}_Q$, where $\hat{X} \in T\mathcal{A}^f_Q$ is any vector which maps onto X under the quotient map $\mu^{-1}(0) \to \mathcal{M}_Q$.

Remark 4.12. Let us assume for the moment that in constructing the hermitian line bundle $\hat{\mathcal{L}}_Q \to \mathcal{A}_Q$ we take the cocycle c_Σ to have, instead of (4.2), the more general form

$$c_\Sigma(\alpha, \lambda) \,=\, \varepsilon_\Sigma([\lambda]) \mathrm{e}^{\pi \mathrm{i} k \int\limits_\Sigma \langle \alpha \wedge \lambda^* \vartheta \rangle} \,,$$

where $[\lambda]$ denotes the homotopy class of the map $\lambda : \Sigma \to \mathbb{T}$. In order to satisfy the cocycle condition (4.3) the $\mathbb{T}$-valued multiplier ε_Σ must have the property

$$\varepsilon_\Sigma([\lambda_1][\lambda_2]) \,=\, \varepsilon_\Sigma([\lambda_1])\, \varepsilon_\Sigma([\lambda_2])\, \mathrm{e}^{\pi \mathrm{i} k \int\limits_\Sigma \langle \lambda_1^* \vartheta \wedge \lambda_2^* \vartheta \rangle}$$

and k can be any integer. All what has been said up to this point, except (4.4 (a)), remains true. We get a hermitian line bundle over $\mathcal{A}_Q$ with a unitary connection with curvature $-2\pi \mathrm{i} k \omega_\Sigma$ and the action of $\mathcal{G}_Q$ lifts preserving the metric and the connection. The restriction of this line to $\mathcal{A}^f_Q$ pushes down to a prequantum line bundle over $\mathcal{M}_Q$. The initial choice (4.2) for c_Σ, in which ε_Σ is taken to be the trivial multiplier (which forces k to be even), is dictated by the requirement to satisfy also the functoriality property (4.4 (a)). This can be seen immediately if we follow the proof given for (4.4 (a)). Only for this choice do we get a prequantum line bundle over the moduli space $\mathcal{M}_\Sigma$ for which there is a lift of the group of orientation preserving diffeomorphisms of Σ. We encountered the same situation in [Ma] in the quantization of symplectic tori, where a prequantum line bundle was picked out by the requirement to have a lift of the group of symmetries of the torus. In the present case the prequantum line bundle over $\mathcal{M}_\Sigma$ picked out by the above stated reasons is also closely connected to the Chern-Simons theory in 3 dimensions. This connection will become clear later in this section.

Given any two trivializable $\mathbb{T}$-bundles Q and Q' over Σ, let $\psi : Q' \to Q$ be a bundle isomorphism. It follows from (4.4 (a)) that the induced isomorphism between the spaces of connections lifts to an isomorphism of line bundles

$$\begin{array}{ccc} \hat{\mathcal{L}}_Q & \xrightarrow{\psi^*} & \hat{\mathcal{L}}_{Q'} \\ \downarrow & & \downarrow \\ \mathcal{A}_Q & \xrightarrow{\psi^*} & \mathcal{A}_{Q'} \end{array} \tag{4.13}$$

To each gauge transformation $\phi \in \mathcal{G}_Q$ there corresponds a gauge transformation $\psi^{-1} \circ \phi \circ \psi \in \mathcal{G}_{Q'}$, so that for any connection $\eta \in \mathcal{A}_Q$ we have the commutative diagram

$$\begin{array}{ccc} L_\eta & \xrightarrow{\psi^*} & L_{\psi^*\eta} \\ {\scriptstyle \phi^*}\downarrow & & \downarrow{\scriptstyle (\psi^{-1}\circ\phi\circ\psi)^*} \\ L_{\phi^*\eta} & \xrightarrow{\psi^*} & L_{\psi^*\phi^*\eta} \end{array} \tag{4.14}$$

of line isomorphisms defined by (4.4(a)). The line bundle isomorphism (4.13) pushes down to an isomorphism between the quotient line bundles by the groups of gauge transformations

$$\begin{array}{ccc} \mathcal{L}_Q = \hat{\mathcal{L}}_Q/\mathcal{G}_Q & \longrightarrow & \mathcal{L}_{Q'} = \hat{\mathcal{L}}_{Q'}/\mathcal{G}_{Q'} \\ \downarrow & & \downarrow \\ \mathcal{A}_Q/\mathcal{G}_Q & \longrightarrow & \mathcal{A}_{Q'}/\mathcal{G}_{Q'} \end{array} \tag{4.15}$$

It follows from (4.14) that the above isomorphism is canonical, that is, independent of the choice of $\mathbb{T}$-bundle isomorphism $\psi : Q' \to Q$. Restricting the above considerations to the subspaces of flat connections we obtain that, for any two trivializable $\mathbb{T}$-bundles Q and Q' over Σ, there is a canonical isomorphism of hermitian line bundles with connections

$$\begin{array}{ccc} \mathcal{L}_Q & \longrightarrow & \mathcal{L}_{Q'} \\ \downarrow & & \downarrow \\ \mathcal{M}_Q & \longrightarrow & \mathcal{M}_{Q'} \end{array} \tag{4.16}$$

Since $\mathcal{M}_\Sigma \cong \mathcal{M}_Q$ for any trivializable $\mathbb{T}$-bundle $Q \to \Sigma$, the above shows that we get a hermitian line bundle

$$\mathcal{L}_\Sigma \to \mathcal{M}_\Sigma$$

with a unitary connection with curvature $-2\pi \mathrm{i} k\omega_\Sigma$, with ω_Σ the standard symplectic form on $\mathcal{M}_\Sigma$.

4.2. **The Chern-Simons section.** We turn now our attention to the 3-dimensional Chern-Simons theory. Let X be a compact oriented 3-manifold with nonempty boundary and $P \to X$ a principal $\mathbb{T}$-bundle with $c_1(P)$ a torsion class. If Θ is a connection on P, then Prop. (3.15) shows that the function

$$\begin{aligned} \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)} : \Gamma(\partial X; Q) &\longrightarrow \mathbb{C} \\ s \qquad &\longmapsto \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)}(s) = \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(s,\Theta)} \end{aligned}$$

is an element of norm 1 in the line $L_{\partial\Theta}$ attached to the restriction of Θ to ∂X

$$\mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)} \in L_{\partial\Theta} \tag{4.17}$$

If $\partial X = \emptyset$, then $\mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)} \in \mathbb{C}$, so we set $L_\emptyset = \mathbb{C}$ to account for this case too.

Theorem 4.18. *Let X be a compact oriented 3-manifold. The assignment*

$$\Theta \longrightarrow e^{\pi i k S_{X,P}(\Theta)} \in L_{\partial\Theta} ,$$

for Θ connections on $\mathbb{T}$-bundles $P \to X$ with $c_1(P)$ torsion, is smooth and satisfies:
(a) Functoriality
Let $\phi : P' \to P$ be a morphism of principal $\mathbb{T}$-bundles covering an orientation preserving diffeomorphism $\bar\phi : X' \to X$. If Θ is a connection on P, then

$$(\partial\phi)^* e^{\pi i k S_{X,P}(\Theta)} = e^{\pi i k S_{X',P'}(\phi^*\Theta)}$$

under the induced isometry $(\partial\phi)^* : L_{\partial\Theta} \to L_{\partial\phi^*\Theta}$. In particular if $\partial X = \emptyset$, then

$$e^{\pi i k S_{X,P}(\Theta)} = e^{\pi i k S_{X',P'}(\phi^*\Theta)}$$

(b) Orientation

If $-X$ is the manifold X with the opposite orientation, then

$$e^{\pi ikS_{-X,P}(\Theta)} = \overline{e^{\pi ikS_{X,P}(\Theta)}} \in \overline{L_{\partial\Theta}}$$

(c) Disjoint union

Let X be the disjoint union $X = X_1 \sqcup X_2$. If Θ_i are connections on $\mathbb{T}$-bundles $P_i \to X_i$, then we have the identification

$$e^{\pi ikS_{X_1\sqcup X_2,P_1\sqcup P_2}(\Theta_1\sqcup\Theta_2)} = e^{\pi ikS_{X_1,P_1}(\Theta_1)} \otimes e^{\pi ikS_{X_2,P_2}(\Theta_2)}$$

under the isomorphism $L_{\partial\Theta_1\sqcup\partial\Theta_2} \cong L_{\partial\Theta_1} \otimes L_{\partial\Theta_2}$.

(d) Gluing

Let X be a compact oriented 3-manifold and let X^{cut} be the 3-manifold obtained by cutting X along a closed oriented 2-dimensional submanifold Σ. Then $\partial X^{cut} = \partial X \sqcup (-\Sigma) \sqcup \Sigma$. Let Θ be a connection on a $\mathbb{T}$-bundle $P \to X$, with η the restriction of Θ to Σ, and let Θ^{cut} denote the induced connection on the pullback bundle $P^{cut} = g^*P$ under the gluing map $g : X^{cut} \to X$. If

$$\mathrm{Tr}_\eta : L_{\partial\Theta^{cut}} \cong L_{\partial\Theta} \otimes \overline{L_\eta} \otimes L_\eta \longrightarrow L_{\partial\Theta}$$

is the contraction map using the hermitian inner product in L_η, then

$$\mathrm{Tr}_\eta\Big(e^{\pi ikS_{X^{cut},P^{cut}}(\Theta^{cut})}\Big) = e^{\pi ikS_{X,P}(\Theta)} \tag{4.19}$$

Proof. (a) follows from (4.4 (a)) and (3.16 (a)).

(b) follows from (4.4 (b)) and (3.16 (b)).

(c) follows from (4.4 (c)) and (3.16 (c)).

(d) Let $g : P^{cut} \to P$ be the bundle map from the pullback bundle $P^{cut} = g^*P$ to P, covering the gluing map $g : X^{cut} \to X$. Then $\Theta^{cut} = g^*\Theta$ and $\partial\Theta^{cut} = \partial\Theta \sqcup \eta \sqcup \eta$. Under the isometry $L_{\partial\Theta^{cut}} \to L_{\partial\Theta} \otimes \overline{L_\eta} \otimes L_\eta$ the element $\mathrm{e}^{\pi \mathrm{i}kS_{X^{cut},P^{cut}}(\Theta^{cut})} \in L_{\partial\Theta^{cut}}$ is mapped to

$$\frac{\mathrm{e}^{\pi\mathrm{i}kS_{X^{cut},P^{cut}}(s^{cut},\Theta^{cut})}}{\mathrm{e}^{\pi\mathrm{i}kS_{X,P}(s,\Theta)}\overline{f(\sigma)}f(\sigma)}\Big(\mathrm{e}^{\pi\mathrm{i}kS_{X,P}(\Theta)} \otimes \bar{f} \otimes f\Big) \in L_{\partial\Theta} \otimes \overline{L_\eta} \otimes L_\eta\,, \tag{4.20}$$

where s, σ and s^{cut} are sections $s : \partial X \to \partial P$, $\sigma : \Sigma \to P\big|_\Sigma$ and $s^{cut} = s \sqcup \sigma \sqcup \sigma : \partial X^{cut} \to \partial P^{cut}$. To prove (4.19) we have to show that

$$\mathrm{e}^{\pi \mathrm{i} k S_{X^{cut},P^{cut}}(s^{cut},\Theta^{cut})} = \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(s,\Theta)}\,. \tag{4.21}$$

Let $\rho_P : P \to \hat{P} = P \times_{\mathbb{T}} SU(2)$ and $\rho_{P^{cut}} : P^{cut} \to \hat{P}^{cut} = P^{cut} \times_{\mathbb{T}} SU(2) = g^*\hat{P}$ be the bundle maps to the induced $SU(2)$-bundles. There is a $SU(2)$-bundle morphism $\hat{g} : \hat{P}^{cut} \to \hat{P}$ so that $\rho_P \circ g = \hat{g} \circ \rho_{P^{cut}}$. Let $\hat{s} : X \to \hat{P}$ be the extension of $\rho_P \circ s : \partial X \to \partial \hat{P}$ and $\hat{s}^{cut} : X \to \hat{P}^{cut}$ the pullback of the section $\hat{s}$. Then $\hat{g} \circ \hat{s}^{cut} = \hat{s} \circ g$ and $\hat{s}^{cut}\big|_{\partial X^{cut}} = \rho_{P^{cut}} \circ s^{cut}$. If $\hat{\Theta}^{cut} = \hat{g}^*\hat{\Theta}$ then $\rho^*_{P^{cut}}\hat{\Theta}^{cut} = \rho_*\Theta^{cut}$, so that $\hat{\Theta}^{cut}$ is the $SU(2)$-connection induced from Θ^{cut}. Therefore, using the definition (3.9) of the Chern-Simons functional, we can write

$$\begin{aligned} S_{X^{cut},P^{cut}}(s^{cut},\Theta^{cut}) &= \int\limits_{X^{cut}} \hat{s}^{cut*}\alpha(\hat{\Theta}^{cut}) \pmod 1 = \int\limits_{X^{cut}} \hat{s}^{cut*}\alpha(\hat{g}^*\hat{\Theta}) \pmod 1 \\ &= \int\limits_{X^{cut}} g^*\hat{s}^*\alpha(\hat{\Theta}) \pmod 1 = \int\limits_X \hat{s}^*\alpha(\hat{\Theta}) \pmod 1 \\ &= S_{X,P}(s,\Theta) \end{aligned}$$

which proves (4.21). Then, taking in (4.20) the hermitian inner product in L_η, i.e. $(f,f) = \overline{f(\sigma)}f(\sigma)$, we get the announced result (4.19). $\square$

Consider now the restriction map $r : \mathcal{A}_P \to \mathcal{A}_{\partial P}$ which sends a connection on the $\mathbb{T}$-bundle $P \to X$ to its restriction over ∂X. As previously shown, to the trivializable $\mathbb{T}$-bundle $\partial P \to \partial X$ there corresponds a hermitian line bundle $\pi : \hat{\mathcal{L}}_{\partial P} \to \mathcal{A}_{\partial P}$ with connection $\hat{\nabla}$ and these pull back under r to a hermitian line bundle $r^*\hat{\mathcal{L}}_{\partial P} \to \mathcal{A}_P$ with connection $r^*\hat{\nabla}$. The restriction of $r^*\hat{\mathcal{L}}_{\partial P}$ to the subspace $\mathcal{A}^f_P$ of flat connections is a *flat* line bundle since

$$(r^*\omega_\Sigma)(\dot{\Theta},\dot{\Theta}') = \omega_\Sigma(r_*\dot{\Theta}, r_*\dot{\Theta}') = -\int\limits_{\partial X} \langle \partial\dot{\Theta} \wedge \partial\dot{\Theta}' \rangle = -\int\limits_X d\langle \dot{\Theta} \wedge \dot{\Theta}' \rangle = 0$$

for any $\dot{\Theta}, \dot{\Theta}' \in T\mathcal{A}^f_P$.

As $\mathcal{G}_P \subset \mathcal{G}_{\partial P}$, the action of the group of gauge transformations $\mathcal{G}_P$ on $\mathcal{A}_P$ lifts to $r^*\hat{\mathcal{L}}_{\partial P}$. Since $r^*\hat{\mathcal{L}}_{\partial P} = \{(\Theta, f) \in \mathcal{A}_P \times \hat{\mathcal{L}}_{\partial P} \mid r(\Theta) = \pi(f)\}$, we see from (4.17)

and (4.18 (a)) that

$$\begin{aligned} \hat{\sigma}_P \,:\, \mathcal{A}_P &\longrightarrow r^*\hat{\mathcal{L}}_{\partial P} \\ \Theta &\longmapsto (\Theta, \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)}) \end{aligned} \tag{4.22}$$

is a nowhere-zero $\mathcal{G}_P$-invariant section, i.e. for any $\phi \in \mathcal{G}_P$,

$$\begin{aligned} \hat{\sigma}_P(\Theta)\cdot\phi \;&=\; (\phi^*\Theta, (\partial\phi)^*\mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)}) \\ &=\; (\phi^*\Theta, \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\phi^*\Theta)}) \;=\; \hat{\sigma}_P(\phi^*\Theta) \end{aligned}$$

Moreover we have:

Proposition 4.23. *The section $\hat{\sigma}_P : \mathcal{A}_P^f \to r^*\hat{\mathcal{L}}_{\partial P}$ is covariantly constant.*

Proof. Let $s : \partial X \to \partial P$ be a section. The induced section $\gamma_s : \mathcal{A}_{\partial P} \to \hat{\mathcal{L}}_{\partial P}$ pulls back to $(r^*\gamma_s)(\Theta) = (\Theta, \gamma_s(\partial\Theta)) : \mathcal{A}_P \to r^*\hat{\mathcal{L}}_{\partial P}$. Since $\mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)} = \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(s,\Theta)}\gamma_s(\partial\Theta)$, we can write $\hat{\sigma}_P = \varphi\,(r^*\gamma_s)$ with $\varphi(\Theta) = \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(s,\Theta)}$. By differentiating φ with respect to $\dot{\eta} \in T_\Theta\mathcal{A}_P^f$, we find

$$\begin{aligned} (\dot{\eta}\varphi)(\Theta) \;&=\; \frac{d}{dt}\Big|_{t=0}\varphi(\Theta + t\dot{\eta}) = \frac{d}{dt}\Big|_{t=0}\mathrm{e}^{\pi \mathrm{i} k S_{X,P}(s,\Theta+t\dot{\eta})} \\ &=\; \frac{d}{dt}\Big|_{t=0}\mathrm{e}^{\pi \mathrm{i} k[S_{X,P}(s,\Theta) - t\int_{\partial X} s^*\langle\Theta\wedge\dot{\eta}\rangle + 2t\int_X\langle d\Theta\wedge\dot{\eta}\rangle + t^2\int_X\langle\dot{\eta}\wedge d\dot{\eta}\rangle]} \\ &=\; \Big[-\pi \mathrm{i} k\int_{\partial X} s^*\langle\Theta\wedge\dot{\eta}\rangle\Big]\,\varphi(\Theta)\,. \end{aligned}$$

On the other hand

$$(r^*\hat{\nabla})_{\dot{\eta}}(r^*\gamma_s) \;=\; -2\pi \mathrm{i} k\,(\dot{\eta} \lrcorner\, r^*\hat{\theta}_s)\,(r^*\gamma_s)\,.$$

Combining these two results we obtain

$$\begin{aligned} (r^*\hat{\nabla})_{\dot{\eta}}\hat{\sigma}_P \;&=\; (r^*\hat{\nabla})_{\dot{\eta}}(\varphi\,(r^*\gamma_s)) \\ &=\; \big[\dot{\eta}(\varphi) - 2\pi \mathrm{i} k(\dot{\eta} \lrcorner\, r^*\hat{\theta}_s)\varphi\big]\,(r^*\gamma_s) \;=\; 0\,. \end{aligned}$$

□

The flat line bundle $r^*\hat{\mathcal{L}}_{\partial P} \to \mathcal{A}_P^f$ together with its metric and connection push down to the quotient by $\mathcal{G}_P$ to a flat hermitian line bundle which is identified with the pullback $r^*\mathcal{L}_{\partial P} \to \mathcal{M}_P$ of the line bundle $\mathcal{L}_{\partial P} \to \mathcal{M}_{\partial P}$ under the restriction

map $r : \mathcal{M}_P \to \mathcal{M}_{\partial P}$. The $\mathcal{G}_P$-invariant and covariantly constant section $\hat{\sigma}_P : \mathcal{A}_P^f \to r^*\hat{\mathcal{L}}_{\partial P}$ corresponds to a covariantly constant section

$$\sigma_P : \mathcal{M}_P \to r^*\mathcal{L}_{\partial P} \,. \tag{4.24}$$

For any two principal $\mathbb{T}$-bundles P and P' over X with $c_1(P) = c_1(P') = p \in \mathrm{Tors}H^2(X;\mathbb{Z})$, let $\phi : P' \to P$ be a bundle isomorphism. This induces an isomorphism of hermitian line bundles with connections $\phi^* : r^*\hat{\mathcal{L}}_{\partial P} \to r^*\hat{\mathcal{L}}_{\partial P'}$ such that for the corresponding Chern-Simons sections (4.22) we have $\phi^*\hat{\sigma}_P(\Theta) = \hat{\sigma}_{P'}(\phi^*\Theta)$, for any $\Theta \in \mathcal{A}_P$. Restricting the line bundles to the subspaces of flat connections and taking the quotient by the group of gauge transformations, we obtain a canonical isomorphism of flat hermitian lines

$$\begin{array}{ccc} r^*\mathcal{L}_{\partial P} & \longrightarrow & r^*\mathcal{L}_{\partial P'} \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}_P & \longrightarrow & \mathcal{M}_{P'} \end{array} \tag{4.25}$$

Thus, for each $p \in \mathrm{Tors}H^2(X;\mathbb{Z})$, we obtain a hermitian line bundle with a flat connection,

$$\mathcal{L}_p \to \mathcal{M}_{X,p} \,,$$

over the connected component $\mathcal{M}_{X,p}$ of the moduli space $\mathcal{M}_X$. The line bundle $\mathcal{L}_p \to \mathcal{M}_{X,p}$ is canonically identified with the restriction to $\mathcal{M}_{X,p} \subset \mathcal{M}_X$ of the pullback of $\mathcal{L}_{\partial X} \to \mathcal{M}_{\partial X}$ under the map $r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}$. In view of the canonical isomorphism (4.25), the section (4.24) defines a nowhere-zero covariantly constant section

$$\sigma_{X,p} \,:\, \mathcal{M}_{X,p} \longrightarrow \mathcal{L}_p = r_X^*\mathcal{L}_{\partial X}\big|_{\mathcal{M}_{X,p}} \tag{4.26}$$

which we refer to as the *Chern-Simons section.*

5. The quantum theory

5.1. **The Hilbert space.** We consider again a closed oriented 2-dimensional manifold Σ and let $g = \frac{1}{2}\dim H^1(\Sigma;\mathbb{R})$. The space of gauge equivalence classes of flat $\mathbb{T}$-connections on Σ is a symplectic $2g$-dimensional manifold $(\mathcal{M}_\Sigma, \omega_\Sigma)$. The construction in Sect.4 provides $(\mathcal{M}_\Sigma, k\omega_\Sigma)$, $k \in 2\mathbb{Z}_+$, with a hermitian line bundle $\mathcal{L}_\Sigma$ with a unitary connection ∇ with curvature $-2\pi\mathrm{i}k\omega_\Sigma$. According to the general geometric quantization scheme the other data needed for the quantization of the symplectic manifold $(\mathcal{M}_\Sigma, k\omega_\Sigma)$ is a polarization.

Since $(\mathcal{M}_\Sigma, \omega_\Sigma)$ is a symplectic torus, we are going to use the results in [Ma] on the quantization with half-densities of symplectic tori in a real polarization. Thus let $L \subset H^1(\Sigma;\mathbb{R})$ be a *rational* Lagrangian subspace of the symplectic vector space $(H^1(\Sigma;\mathbb{R}), \omega_\Sigma)$. The attribute rational refers to the fact that the intersection $L \cap H^1(\Sigma;\mathbb{Z})$ with the integer lattice $H^1(\Sigma;\mathbb{Z}) \subset H^1(\Sigma;\mathbb{R})$ generates L as a vector space. Under the identification of the tangent space at any point of $\mathcal{M}_\Sigma$ with $2\pi\mathrm{i}H^1(\Sigma;\mathbb{R})$, the Lagrangian subspace L determines an invariant real polarization $\mathcal{P}_L$ of $(\mathcal{M}_\Sigma, \omega_\Sigma)$. The polarization $\mathcal{P}_L$ is the tangent bundle along the leaves of an invariant Lagrangian foliation of $\mathcal{M}_\Sigma$.

The bundle of half-densities on $\mathcal{P}_L$ is the line bundle $|\mathrm{Det}\mathcal{P}_L^*|^{\frac{1}{2}}$ over $\mathcal{M}_\Sigma$. Its restriction to any leaf Λ of $\mathcal{P}_L$ has a canonical flat connection $\nabla^{\mathcal{P}_L}$ defined as follows [Wo]. First let us recall that, since the quotient map $\pi : \mathcal{M}_\Sigma \to \mathcal{M}_\Sigma/\mathcal{P}_L$ is a smooth fibration, for each point in $\mathcal{M}_\Sigma/\mathcal{P}_L$ there is a local neighborhood U such that $\mathcal{P}_L\big|_{\pi^{-1}(U)}$ is spanned by Hamiltonian vector fields $[\dot{x}_1], \dots, [\dot{x}_g]$. Then, on $\pi^{-1}(U)$, the covariant derivative $\nabla^{\mathcal{P}_L}_{[\dot{w}]}\mu$ of a section μ of $|\mathrm{Det}\mathcal{P}_L^*|^{\frac{1}{2}}$, along a vector field $[\dot{w}] \in \mathcal{P}_L\big|_{\pi^{-1}(U)}$, is defined by

$$\left(\nabla^{\mathcal{P}_L}_{[\dot{w}]}\mu\right)([\dot{x}_1], \dots, [\dot{x}_g]) \;=\; [\dot{w}]\big(\mu([\dot{x}_1], \dots, [\dot{x}_g])\big)\,.$$

Since the leaves of $\mathcal{P}_L$ are diffeomorphic to g-dimensional tori, the distribution $\mathcal{P}_L$ has a canonical density κ invariant under the Hamiltonian vector fields in $\mathcal{P}_L$ and which assigns to each integral manifold of $\mathcal{P}_L$ the volume 1. The square root

of κ trivializes the line bundle $|\mathrm{Det}\mathcal{P}_L^*|^{\frac{1}{2}}$ and is a covariantly constant section of $\big(|\mathrm{Det}\mathcal{P}_L^*|^{\frac{1}{2}}\big)\big|_\Lambda$ for each leaf Λ of $\mathcal{P}_L$.

The Hilbert space of quantization for $(\mathcal{M}_\Sigma, k\omega_\Sigma)$ will be defined in terms of sections of the line bundle $\mathcal{L}_\Sigma \otimes |\mathrm{Det}\mathcal{P}_L^*|^{\frac{1}{2}}$ obtained by tensoring the prequantum line bundle $\mathcal{L}_\Sigma$ with the bundle of half-densities on $\mathcal{P}_L$. The restriction of this line bundle to a leaf Λ of $\mathcal{P}_L$ has a flat connection defined by

$$\nabla_{[\dot{w}]}(\sigma \otimes \mu) \;=\; \nabla_{[\dot{w}]}\sigma \otimes \mu + \sigma \otimes \nabla^{\mathcal{P}_L}_{[\dot{w}]}\mu\,,$$

for any $[\dot{w}] \in \mathcal{P}_L\big|_\Lambda$ and $\sigma \otimes \mu \in \Gamma(\Lambda; \mathcal{L}_\Sigma \otimes |\mathrm{Det}\mathcal{P}_L^*|^{\frac{1}{2}})$. The union of those leaves of $\mathcal{P}_L$ for which the holonomy group of this connection is trivial defines the Bohr-Sommerfeld set $\mathcal{BS}_{\mathcal{P}_L}$ of the polarization $\mathcal{P}_L$. For each leaf Λ belonging to $\mathcal{BS}_{\mathcal{P}_L}$ we let S_Λ denote the one-dimensional vector space of covariantly constant sections of $\big(\mathcal{L}_\Sigma \otimes |\mathrm{Det}\mathcal{P}_L^*|^{\frac{1}{2}}\big)\big|_\Lambda$. Then the Hilbert space associated to the symplectic manifold $(\mathcal{M}_\Sigma, k\omega_\Sigma)$ with real polarization $\mathcal{P}_L$ is defined to be the complex vector space

$$\mathcal{H}(\Sigma, L) \;=\; \bigoplus_{\Lambda \subset \mathcal{BS}_{\mathcal{P}_L}} S_\Lambda$$

with inner product

$$\langle \sigma \otimes \mu, \sigma' \otimes \mu' \rangle \;=\; \begin{cases} 0\,, & \text{if } \sigma \otimes \mu \in S_\Lambda,\ \sigma' \otimes \mu' \in S_{\Lambda'},\ \Lambda \neq \Lambda' \\ \int_\Lambda (\sigma, \sigma')\mu * \mu'\,, & \text{if } \sigma \otimes \mu, \sigma' \otimes \mu' \in S_\Lambda \end{cases}$$

(σ, σ') is the function on Λ obtained by taking the hermitian inner product in the fibre of $\mathcal{L}_\Sigma$ and $\mu * \mu'$ the density on Λ defined by

$$(\mu * \mu')([\dot{x}_1], \ldots, [\dot{x}_g]) \;=\; \mu([\dot{x}_1], \ldots, [\dot{x}_g])\, \mu'([\dot{x}_1], \ldots, [\dot{x}_g])\,,$$

for some vector fields $[\dot{x}_1], \ldots, [\dot{x}_g]$ spanning $T\Lambda$.

Now recall that we identify $\mathcal{M}_\Sigma = \frac{2\pi\mathrm{i}H^1(\Sigma;\mathbb{R})}{2\pi\mathrm{i}H^1(\Sigma;\mathbb{Z})}$. Let $[\dot{w}_1], \ldots, [\dot{w}_g]$ be a basis for $2\pi\mathrm{i}(L \cap H^1(\Sigma;\mathbb{Z}))$. Then, as shown in ([Ma], §3), the Bohr-Sommerfeld leaves of $\mathcal{P}_L$ on $\mathcal{M}_\Sigma$ are determined by the requirement that their preimages in the linear symplectic space $2\pi\mathrm{i}H^1(\Sigma;\mathbb{R})$ covering $\mathcal{M}_\Sigma$ satisfy the condition

$$\mathrm{e}^{2\pi\mathrm{i}k\omega_\Sigma([\dot{w}_i],[x])} \;=\; 1\,, \qquad i = 1, \ldots, g\,. \tag{5.1}$$

For each $\mathbf{q} = (q_1, \ldots, q_g) \in (\mathbb{Z}/k\mathbb{Z})^g$ the linear equations

$$k\, \omega_\Sigma([\dot{w}_i], [x]) \;=\; q_i \pmod{k}\,, \qquad i = 1, \ldots, g\,, \tag{5.2}$$

for $[x] \in 2\pi \mathrm{i} H^1(\Sigma; \mathbb{R})$, define a family of parallel Lagrangian planes projecting onto a Bohr-Sommerfeld leaf $\Lambda_{\mathbf{q}}$ under the quotient map $2\pi \mathrm{i} H^1(\Sigma; \mathbb{R}) \to \mathcal{M}_\Sigma$. Thus the dimension of the Hilbert space $\mathcal{H}(\Sigma, L)$ is k^g.

In conclusion, to each closed oriented surface Σ together with a rational Lagrangian subspace $L \subset H^1(\Sigma; \mathbb{R})$ we associate by the previous construction a finite dimensional Hilbert space $\mathcal{H}(\Sigma, L)$. We recall the following results from [Ma].

If $L_1, L_2 \subset H^1(\Sigma; \mathbb{R})$ are two rational Lagrangian subspaces, then there is a canonical unitary operator

$$F_{L_2 L_1} \,:\, \mathcal{H}(\Sigma, L_1) \longrightarrow \mathcal{H}(\Sigma, L_2) \tag{5.3}$$

induced by the Blattner-Kostant-Sternberg (BKS) pairing

$$\langle\langle \cdot, \cdot \rangle\rangle : \mathcal{H}(\Sigma, L_2) \times \mathcal{H}(\Sigma, L_1) \longrightarrow \mathbb{C}$$

The BKS pairing between the Hilbert spaces $\mathcal{H}(\Sigma, L_1) = \bigoplus_{\Lambda_1 \subset \mathcal{BS}_{\mathcal{P}_{L_1}}} S_{\Lambda_1}$ and $\mathcal{H}(\Sigma, L_2) = \bigoplus_{\Lambda_2 \subset \mathcal{BS}_{\mathcal{P}_{L_2}}} S_{\Lambda_2}$ is defined by setting

$$\langle\langle s_2 \otimes \mu_2, s_1 \otimes \mu_1 \rangle\rangle \;=\; \int\limits_{\Lambda_1 \cap \Lambda_2} (s_2, s_1)\; \mu_2 * \mu_1\,, \tag{5.4}$$

for any Bohr-Sommerfeld leaves $\Lambda_1 \subset \mathcal{BS}_{\mathcal{P}_{L_1}}$, $\Lambda_2 \subset \mathcal{BS}_{\mathcal{P}_{L_2}}$ and for any sections $s_1 \otimes \mu_1 \in S_{\Lambda_1}$, $s_2 \otimes \mu_2 \in S_{\Lambda_2}$. The density $\mu_2 * \mu_1$ on $\Lambda_1 \cap \Lambda_2$ is defined as follows [Sn]. For any point $[x] \in \Lambda_1 \cap \Lambda_2$, choose a symplectic basis $([\dot{\boldsymbol{v}}_{\mathbf{2}}], [\dot{\boldsymbol{w}}]; [\dot{\boldsymbol{v}}_{\mathbf{1}}], [\dot{\boldsymbol{t}}])$ for $(\mathcal{M}_\Sigma, k\omega_\Sigma)$, that is a basis satisfying $k\omega_\Sigma([\dot{w}_i], [\dot{t}_j]) \;=\; \delta_{ij}$, $k\omega_\Sigma([\dot{v}_{2i}], [\dot{v}_{1j}]) \;=\; \delta_{ij}$ and $k\omega_\Sigma([\dot{v}_{1i}], [\dot{t}_j]) \;=\; k\omega_\Sigma([\dot{v}_{2i}], [\dot{t}_j]) = 0$, and with $[\dot{\boldsymbol{w}}]$ a basis for $T_{[x]}(\Lambda_1 \cap \Lambda_2)$, $([\dot{\boldsymbol{v}}_{\mathbf{1}}], [\dot{\boldsymbol{w}}])$ a basis for $T_{[x]}\Lambda_1$ and $([\dot{\boldsymbol{v}}_{\mathbf{2}}], [\dot{\boldsymbol{w}}])$ a basis for $T_{[x]}\Lambda_2$. Then $\mu_2 * \mu_1$ is the density on $\Lambda_1 \cap \Lambda_2$ defined by

$$(\mu_2 * \mu_1)([\dot{\boldsymbol{w}}]) \;=\; \mu_2([\dot{\boldsymbol{v}}_{\mathbf{2}}], [\dot{\boldsymbol{w}}])\, \mu_1([\dot{\boldsymbol{v}}_{\mathbf{1}}], [\dot{\boldsymbol{w}}]).$$

The operator $F_{L_2L_1}$ is determined by the relation

$$\langle s_2 \otimes \mu_2, F_{L_2L_1}(s_1 \otimes \mu_1)\rangle \;=\; \langle\langle s_2 \otimes \mu_2, s_1 \otimes \mu_1\rangle\rangle\,. \tag{5.5}$$

For the unitarity proof of $F_{L_2L_1}$ we refer to ([Ma],§4).

For any three rational Lagrangian subspaces $L_1, L_2, L_3 \subset H^1(\Sigma;\mathbb{R})$ the unitary operators relating the Hilbert spaces associated to Σ and each of these Lagrangian subspaces compose transitively up to a projective factor expressible in terms of the Maslov-Kashiwara index $\tau(L_1, L_2, L_3)$:

$$F_{L_1L_3} \circ F_{L_3L_2} \circ F_{L_2L_1} \;=\; \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}\tau(L_1,L_2,L_3)}\, I \tag{5.6}$$

For the proof of the above relation we refer to ([Ma],§6). The definition of the index τ may be found in [Go, LV, Ma].

5.2. **The vector.** We consider a compact oriented 3-manifold X with nonempty boundary ∂X. Through the construction of Sect.4 we have a prequantum line bundle $\mathcal{L}_{\partial X}$ on the symplectic space $(\mathcal{M}_{\partial X}, k\omega_{\partial X})$. The Lagrangian map $r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}$ determined by the restriction of connections on X to ∂X suggests a natural choice for the polarization needed in quantizing $(\mathcal{M}_{\partial X}, k\omega_{\partial X})$. That is, we take the invariant real polarization $\mathcal{P}_X$ on $\mathcal{M}_{\partial X}$ determined by the rational Lagrangian subspace $L_X \subset H^1(\partial X;\mathbb{R})$, where $L_X \;=\; \mathrm{Im}\{\dot{r}_X : H^1(X;\mathbb{R}) \to H^1(\partial X;\mathbb{R})\}$. To the pair $(\partial X, L_X)$ corresponds the Hilbert space

$$\mathcal{H}(\partial X, L_X) \;=\; \bigoplus_{\Lambda \subset \mathcal{BS}_{\mathcal{P}_X}} S_\Lambda$$

defined as the direct sum of the one-dimensional vector spaces S_Λ of parallel sections of the line bundle $\mathcal{L}_{\partial X} \otimes |\mathrm{Det}\mathcal{P}_X^*|^{\frac{1}{2}}$, supported on the leaves Λ contained in the Bohr-Sommerfeld set $\mathcal{BS}_{\mathcal{P}_X}$ of the polarization $\mathcal{P}_X$.

Our goal is to give a canonical construction of a vector Z_X in $\mathcal{H}(\partial X, L_X)$ to be associated to the 3-manifold X. First we note the following

Proposition 5.7. *The Lagrangian submanifold* $\Lambda_X = \mathrm{Im}\{r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}\}$ *of* $\mathcal{M}_{\partial X}$ *is contained in the Bohr-Sommerfeld set* $\mathcal{BS}_{\mathcal{P}_X}$ *and is connected.*

Proof. We start with the observation that for any flat connection Θ on a $\mathbb{T}$-bundle $P \to X$ and any closed 1-form $\dot\alpha$ on X for which the cohomology class $\frac{1}{2\pi\mathrm{i}}[\dot\alpha]$ in $H^1(X;\mathbb{R})$ is integral we have

$$\int\limits_{\partial X} \langle s^*\Theta \wedge \dot\alpha\rangle \;=\; 0 \quad (\mathrm{mod}\ 1)\,,$$

for any section $s : \partial X \to \partial P$. This follows from (3.20), since $F_\Theta = 0$ and since any integral class $[\dot\alpha]$ in $2\pi\mathrm{i}H^1(X;\mathbb{Z})$ is the cohomology class $[u^*\vartheta]$ of the pullback of the Maurer-Cartan form ϑ of the group $\mathbb{T}$ through some map $u : X \to \mathbb{T}$. According to the definition (2.4) of the symplectic form $\omega_{\partial X}$ on $\mathcal{M}_{\partial X} = \frac{2\pi\mathrm{i}H^1(\partial X;\mathbb{R})}{2\pi\mathrm{i}H^1(\partial X;\mathbb{Z})}$, we can rewrite the above equation as

$$\omega_{\partial X}(\dot r_X[\dot\alpha], [s^*\Theta]) \;=\; 0 \quad (\mathrm{mod}\ 1)\,, \tag{5.8}$$

with $[s^*\Theta] \in 2\pi\mathrm{i}H^1(\partial X;\mathbb{R})$ the cohomology class of the 1-form $s^*\Theta$ on ∂X and $\dot r_X[\dot\alpha]$ the image of $[\dot\alpha]$ under the restriction map $\dot r_X : H^1(X;\mathbb{R}) \to H^1(\partial X;\mathbb{R})$.

Let $g = \frac{1}{2}\dim H^1(\partial X;\mathbb{R})$ and choose a basis $[\dot w_1], \ldots, [\dot w_g]$ for the subspace $2\pi\mathrm{i}(L_X \cap H^1(\partial X;\mathbb{Z}))$ of $2\pi\mathrm{i}H^1(\partial X;\mathbb{R})$, with $[\dot w_i] = \dot r_X[\dot\alpha_i]$ for some integral classes $[\dot\alpha_i] \in 2\pi\mathrm{i}H^1(X;\mathbb{Z})$. Given any $[\eta] \in \Lambda_X$, there exists $[\Theta] \in \mathcal{M}_X$ with Θ a flat connection on a $\mathbb{T}$-bundle $P \to X$ such that its restriction to ∂X is $\partial\Theta = \eta$. For any section $s : \partial X \to \partial P$, the element $[s^*\eta] \in 2\pi\mathrm{i}H^1(\partial X;\mathbb{R})$ projects to $[\eta]$ under the quotient map $2\pi\mathrm{i}H^1(\partial X;\mathbb{R}) \to \mathcal{M}_{\partial X} = \frac{2\pi\mathrm{i}H^1(\partial X;\mathbb{R})}{2\pi\mathrm{i}H^1(\partial X;\mathbb{Z})}$. Then it follows from (5.8) that

$$\omega_{\partial X}([\dot w_i], [s^*\eta]) \;=\; \omega_{\partial X}(\dot r_X[\dot\alpha_i], [s^*\Theta]) \;=\; 0 \quad (\mathrm{mod}\ 1)\,, \quad i = 1, \ldots, g\,.$$

Now, since Λ_X is a Lagrangian submanifold of $(\mathcal{M}_{\partial X}, \omega_{\partial X})$ and since $T\Lambda_X = \mathcal{P}_X\big|_{\Lambda_X}$, each connected component of Λ_X is a leaf of $\mathcal{P}_X$. A comparison of the above equation to the Bohr-Sommerfeld condition (5.2) shows that the leaves of the polarization $\mathcal{P}_X$ contained in Λ_X are all Bohr-Sommerfeld leaves. Moreover, it shows that Λ_X contains only one Bohr-Sommerfeld leaf, that is, Λ_X is connected. $\square$

The above proposition suggests that we might try to define the vector Z_X in $\mathcal{H}(\partial X, L_X)$ as a parallel section of the line bundle $\mathcal{L}_{\partial X} \otimes |\mathrm{Det}\mathcal{P}_X^*|^{\frac{1}{2}}$ restricted to the Bohr-Sommerfeld leaf Λ_X. First we note that, since Λ_X is a connected Lagrangian submanifold of $\mathcal{M}_{\partial X}$, each connected component $\mathcal{M}_{X,p}$ of the moduli space $\mathcal{M}_X = \bigsqcup_{p \in \mathrm{Tors} H^2(X;\mathbb{Z})} \mathcal{M}_{X,p}$ maps *onto* Λ_X under the continuous map $r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}$. Now recall from Sect.4 that, for each component $\mathcal{M}_{X,p}$, the restriction $\mathcal{L}_p = r_X^* \mathcal{L}_{\partial X}\big|_{\mathcal{M}_{X,p}}$ of the pullback under r_X of the prequantum line bundle $\mathcal{L}_{\partial X} \to \mathcal{M}_{\partial X}$ has a nowhere-zero parallel section, the Chern-Simons section $\sigma_{X,p}$. Then, since Λ_X is a Bohr-Sommerfeld leaf and since the map $r_X : \mathcal{M}_{X,p} \to \Lambda_X$ is surjective, it follows that the section $\sigma_{X,p}$ is the pullback of a nowhere-zero covariantly constant section of $\mathcal{L}_{\partial X}\big|_{\Lambda_X}$ which we continue to denote by $\sigma_{X,p}$. We are thus led to define a section σ_X of $\mathcal{L}_{\partial X}\big|_{\Lambda_X}$ by

$$\sigma_X = \sum_{p \in \mathrm{Tors} H^2(X;\mathbb{Z})} \sigma_{X,p} \,. \tag{5.9}$$

σ_X is covariantly constant with respect to the connection ∇ in $\mathcal{L}_{\partial X}$.

What we need now is a covariantly constant global section μ_X of the bundle of half-densities $|\mathrm{Det}\mathcal{P}_X^*|^{\frac{1}{2}}$ over Λ_X. We construct such a section using the Reidemeister torsion (R-torsion) invariant T_X of the 3-manifold X.

The R-torsion T_X of a compact manifold X is defined as a norm on the determinant line $|\mathrm{Det} H^\bullet(X;\mathbb{R})|$ of the cohomology of X. We recall that $|\mathrm{Det} H^\bullet(X;\mathbb{R})| = \otimes_q |\mathrm{Det} H^q(X;\mathbb{R})|^{(-1)^{q+1}}$. The definition and properties of the R-torsion may be found in [RS1, RS2, Mü, V]. The R-torsion T_X is defined in terms of a smooth triangulation K of X. Let $(C^\bullet(K), d^c)$ be the cochain complex of K. There is a preferred basis $\mathbf{c}^{(q)} = (c_i^{(q)})$ of $C^q(K)$ defined by $c_i^{(q)}(\sigma_j^{(q)}) = \delta_{ij}$, where $(\sigma_j^{(q)})$ are the q-simplices of K. The space $C^q(K)$ has a natural inner product for which the basic cochains $(c_i^{(q)})$ are orthonormal. This inner product passes to the cohomology space $H^q(X;\mathbb{R}) = H^q(C^\bullet(K))$. Let $\mathbf{h}^{(q)} = (h_1^{(q)}, \ldots, h_{\beta_q}^{(q)})$, with $\beta_q = \dim H^q(X;\mathbb{R})$, be a basis for $H^q(C^\bullet(K))$ and let $\tilde{h}_i^{(q)}$ be representative cocycles in $C^q(K)$ for the elements $h_i^{(q)}$. For each q, choose a basis $\mathbf{b}^{(q)}$ for the image $d^c C^{q-1}(K) \subset C^q(K)$ and let $\tilde{\mathbf{b}}^{(q)}$ be an independent set in $C^{q-1}(K)$ such that $d^c \tilde{\mathbf{b}}^{(q)} = \mathbf{b}^{(q)}$. Then

$(\mathbf{b}^{(q)}, \tilde{\mathbf{b}}^{(q+1)}, \tilde{\mathbf{h}}^{(q)})$ is a basis for $C^q(K)$. Let D_q denote the matrix representing the change of basis from $(\mathbf{b}^{(q)}, \tilde{\mathbf{b}}^{(q+1)}, \tilde{\mathbf{h}}^{(q)})$ to $\mathbf{c}^{(q)}$. The determinant of D_q depends only on the choice of $\mathbf{b}^{(q)}, \mathbf{b}^{(q+1)}, \mathbf{h}^{(q)}$. The R-torsion of X is the density $T_X \in |\mathrm{Det} H^\bullet(X;\mathbb{R})^*|$ defined by

$$T_X \;=\; \bigotimes_{q=0}^{\dim X} \left|(\det D_q)^{-1}\,(h_1^{(q)} \wedge \cdots \wedge h_{\beta_q}^{(q)})\right|^{(-1)^q} \tag{5.10}$$

The above expression is independent of the choice of bases $\mathbf{b}^{(q)}$ and $\mathbf{h}^{(q)}$. The norm (5.10) is known to be a combinatorial invariant of K; hence any smooth triangulation of the compact manifold X gives the same R-torsion norm on $|\mathrm{Det} H^\bullet(X;\mathbb{R})|$.

For a compact oriented 3-manifold X with nonempty boundary the R-torsion T_X belongs to $\big|\mathrm{Det} H^0(X;\mathbb{R})\big|\otimes\big|\mathrm{Det} H^1(X;\mathbb{R})^*\big|\otimes\big|\mathrm{Det} H^2(X;\mathbb{R})\big|\otimes\big|\mathrm{Det} H^3(X;\mathbb{R})^*\big|$. By Poincare duality $H^2(X;\mathbb{R}) \cong H^1(X,\partial X;\mathbb{R})^*$. Choosing canonical trivializations $\big|\mathrm{Det} H^0(X;\mathbb{R})\big| \cong \mathbb{R}_+$ and $\big|\mathrm{Det} H^3(X;\mathbb{R})\big| \cong \mathbb{R}_+$ with respect to orthonormal bases for $H^0(X;\mathbb{R})$ and $H^3(X;\mathbb{R})$, we can then regard T_X as a density

$$T_X \;\in\; \big|\mathrm{Det} H^1(X;\mathbb{R})^*\big| \otimes \big|\mathrm{Det} H^1(X,\partial X;\mathbb{R})^*\big| \tag{5.11}$$

We recall from Sect.2 that $\mathcal{M}_X = H^1(X;\mathbb{T})$ and that we have an exact sequence (2.8) of compact abelian Lie groups. The corresponding sequence between the tangent spaces at the identity of the groups in (2.8),

$$\begin{aligned} 0 \to H^0(X,\partial X;\mathbb{R}) \to H^0(X;\mathbb{R}) \to H^0(\partial X;\mathbb{R}) \to& \\ \to H^1(X,\partial X;\mathbb{R}) \to H^1(X;\mathbb{R}) \to T_e\Lambda_X \to 0\,,& \end{aligned}$$

induces the following isomorphism between the half-densities spaces:

$$|\mathrm{Det} T_e^*\Lambda_X|^{\frac{1}{2}} \cong \big|\mathrm{Det} H^1(X;\mathbb{R})^*\big|^{\frac{1}{2}} \otimes \big|\mathrm{Det} H^1(X,\partial X;\mathbb{R})\big|^{\frac{1}{2}}\ , \tag{5.12}$$

where we used again canonical trivializations for the spaces $H^0(*;\mathbb{R})$. Let us pick an arbitrary element $w \in \big|\mathrm{Det} H^1(X,\partial X;\mathbb{R})^*\big|$. Since $H^1(X,\partial X;\mathbb{R})$ is identified with the tangent space at the identity of the group $H^1(X,\partial X;\mathbb{T})$, w extends to an invariant density $\mathbf{w}$ on $H^1(X,\partial X;\mathbb{T})$. If $w^{-1} \in \big|\mathrm{Det} H^1(X,\partial X;\mathbb{R})\big|$ stands for the dual of w, we note that $(T_X)^{\frac{1}{2}} \otimes w^{-1} \in \big|\mathrm{Det} H^1(X;\mathbb{R})^*\big|^{\frac{1}{2}} \otimes \big|\mathrm{Det} H^1(X,\partial X;\mathbb{R})\big|^{\frac{1}{2}}$.

Thus we can define an invariant half-density μ_X on the abelian group Λ_X, whose value at the identity is

$$(\mu_X)_e = \left[\int_{H^1(X,\partial X;\mathbb{T})} \mathbf{w} \right] (T_X)^{\frac{1}{2}} \otimes w^{-1} \tag{5.13}$$

The identification between the l.h.s. and r.h.s. of the above equation is made via the isomorphism (5.12). As $T\Lambda_X = \mathcal{P}_X\big|_{\Lambda_X}$, the half-density μ_X, being Λ_X-invariant, defines therefore a covariantly constant section of the half-densities bundle $\left(|\mathrm{Det}\mathcal{P}_X^*|^{\frac{1}{2}}\right)\big|_{\Lambda_X}$. We introduce the notation

$$\mu_X = \int_{H^1(X,\partial X;\mathbb{T})} (T_X)^{\frac{1}{2}} \tag{5.14}$$

with the implicit understanding of μ_X as the invariant extension of (5.13).

Using (5.9) and (5.14) we associate then to the compact oriented 3-manifold X with boundary ∂X the vector Z_X in the Hilbert space $\mathcal{H}(\partial X, L_X)$, defined by the expression

$$\begin{aligned} Z_X &= \frac{k^{m_X}}{[\#\mathrm{Tors}H^2(X;\mathbb{Z})]}\, \sigma_X \otimes \mu_X \\ &= \frac{k^{m_X}}{[\#\mathrm{Tors}H^2(X;\mathbb{Z})]} \sum_{p\in \mathrm{Tors}H^2(X;\mathbb{Z})} \sigma_{X,p} \otimes \int_{H^1(X,\partial X;\mathbb{T})} (T_X)^{\frac{1}{2}} \end{aligned} \tag{5.15}$$

where

$$\begin{aligned} m_X = \frac{1}{4} \big(&\dim H^1(X;\mathbb{R}) + \dim H^1(X,\partial X;\mathbb{R}) \\ &- \dim H^0(X;\mathbb{R}) - \dim H^0(X,\partial X;\mathbb{R}) \big)\,. \end{aligned} \tag{5.16}$$

If X is a closed oriented 3-manifold then, according to (5.11), the square root of the R-torsion T_X is a density $(T_X)^{\frac{1}{2}} \in \left|\mathrm{Det}H^1(X;\mathbb{R})^*\right|$. Since the tangent space $T\mathcal{M}_X \cong H^1(X;\mathbb{R})$, this defines an invariant density $(T_X)^{\frac{1}{2}}$ on the moduli space $\mathcal{M}_X = H^1(X;\mathbb{T})$. From (4.18 (a)) it follows that we have a well-defined function σ_X on $\mathcal{M}_X$ whose restriction to each connected component $\mathcal{M}_{X,p}$ of $\mathcal{M}_X$ is given by $\sigma_{X,p}([\Theta]) = \mathrm{e}^{\pi i k S_{X,P}(\Theta)}$, for any $\mathbb{T}$-bundle P with $c_1(P) = p$. As a result of

(3.8) σ_X is constant on components. Thus to the closed oriented 3-manifold X we associate the complex number

$$
\begin{aligned}
Z_X &= \frac{k^{m_X}}{[\#\mathrm{Tors} H^2(X;\mathbb{Z})]} \sum_{p \in \mathrm{Tors} H^2(X;\mathbb{Z})} \sigma_{X,p} \int\limits_{\mathcal{M}_X} (T_X)^{\frac{1}{2}} \\
&= k^{m_X} \sum_{p \in \mathrm{Tors} H^2(X;\mathbb{R})} \int\limits_{\mathcal{M}_{X,p}} \sigma_{X,p}\,(T_X)^{\frac{1}{2}} \\
&= k^{m_X} \int\limits_{\mathcal{M}_X} \sigma_X\,(T_X)^{\frac{1}{2}}
\end{aligned} \tag{5.17}
$$

where

$$
m_X = \frac{1}{2}\left(\dim H^1(X;\mathbb{R}) - \dim H^0(X;\mathbb{R})\right). \tag{5.18}
$$

6. The topological quantum field theory

The results of the previous section show that we are very close to defining a 2+1 dimensional topological quantum field theory (TQFT) for the Lie group $\mathbb{T}$ and an even integer k. At this stage we know how to canonically associate to a closed oriented 2-manifold Σ together with a rational Lagrangian subspace $L \subset H^1(\Sigma;\mathbb{R})$ a finite dimensional Hilbert space $\mathcal{H}(\Sigma, L)$ and to a compact oriented 3-manifold X a vector Z_X in the Hilbert space $\mathcal{H}(\partial X, L_X)$ associated to ∂X and the Lagrangian subspace $L_X \subset H^1(\partial X;\mathbb{R})$ determined by X.

A TQFT is supposed to satisfy the axioms listed in [A1] which refer to functoriality, the orientation and disjoint union properties and the gluing of manifolds along their common boundaries.

In order to construct the Chern-Simons TQFT for the group $\mathbb{T}$ we are going to use the notions introduced in [Wa] of extended 2- and 3-manifolds, extended morphisms and gluing of extended 3-manifolds. We give below the relevant definitions. As in [Wa] we use the shortened notation with the prefix 'e-' standing for 'extended'.

Definition 6.1. An *e-2-manifold* is a pair (Σ, L), with Σ a closed oriented 2-dimensional manifold and L a rational Lagrangian subspace of $H^1(\Sigma;\mathbb{R})$.

Definition 6.2. (i) An *e-3-manifold* is a triple (X, L, n), with X a compact oriented 3-manifold, L a rational Lagrangian subspace of $H^1(\partial X;\mathbb{R})$ and $n \in \mathbb{Z}/8\mathbb{Z}$.
(ii) The boundary of an e-3-manifold is $\partial(X, L, n) = (\partial X, L)$.
A closed e-3-manifold is just a pair (X, n) with X a closed oriented 3-manifold and $n \in \mathbb{Z}/8\mathbb{Z}$.

Definition 6.3. An extended morphism from an e-2-manifold (Σ', L') to an e-2-manifold (Σ, L) is a pair (h, m) with $h : \Sigma' \to \Sigma$ an orientation preserving diffeomorphism and $m \in \mathbb{Z}/8\mathbb{Z}$. We refer to extended morphisms between e-2-manifolds as *e-2-morphisms.*

Definition 6.4. *Composition of e-2-morphisms.*
If $(h, m) : (\Sigma', L') \to (\Sigma, L)$ and $(h', m') : (\Sigma'', L'') \to (\Sigma', L')$ are e-2-morphisms, their composition is defined to be

$$(h, m)(h', m') = \big(hh',\; m + m' + \tau(L'', h'^* L', (hh')^* L) \pmod 8\big)\,,$$

where τ is the Maslov-Kashiwara index [Go, LV] of a triple of Lagrangian subspaces of a symplectic vector space.

The e-2-morphism $(id, 0)$ acts as the identity and the inverse of an e-2-morphism (h, m) is $(h, m)^{-1} = (h^{-1}, -m)$.

Definition 6.5. An extended morphism from an e-3-manifold (X, L, n) to an e-3-manifold (X', L', n') is a pair (Φ, m) with $\Phi : X' \to X$ an orientation preserving diffeomorphism and $m \in \mathbb{Z}/8\mathbb{Z}$ such that

$$n' = n + m + \tau(L_{X'}, L', (\partial\Phi)^* L) \pmod 8$$

where $\partial\Phi : \partial X' \to \partial X$ is the induced diffeomorphism between the boundaries. We refer to extended morphisms between e-3-manifolds as *e-3-morphisms.*

To the e-3-morphism $(\Phi, m) : (X', L', n') \to (X, L, n)$ there corresponds an induced e-2-morphism between the boundaries $(\partial\Phi, m) : (\partial X', L') \to (\partial X, L)$.
Two closed e-3-manifolds (X, n) and (X', n') are isomorphic if there exists an orientation preserving diffeomorphism $\Phi : X' \to X$ and $n' = n$.

Definition 6.6. *Composition of e-3-morphisms.*
If $(\Phi, m) : (X', L', n') \to (X, L, n)$ and $(\Phi', m') : (X'', L'', n'') \to (X', L', n')$ are e-3-morphisms, then their composition is the e-3-morphism

$$(\Phi, m)(\Phi', m') = \big(\Phi\Phi',\, m + m' + \tau(L'', (\partial\Phi')^* L', (\partial\Phi\partial\Phi')^* L) \pmod 8\big)$$

From the cocycle and the symplectic invariance properties [LV, Go] of the Maslov-Kashiwara index it follows that the equality

$$\begin{aligned} n'' = n + m + m' + \tau(L'', (\partial\Phi')^* L', (\partial\Phi\partial\Phi')^* L) + \\ + \tau(L_{X''}, L'', (\partial\Phi\partial\Phi')^* L) \pmod 8 \end{aligned}$$

is indeed satisfied.

Definition 6.7. (i) If (Σ_1, L_1) and (Σ_2, L_2) are e-2-manifolds, their disjoint union is the e-2-manifold

$$(\Sigma_1, L_1) \sqcup (\Sigma_2, L_2) \;=\; (\Sigma_1 \sqcup \Sigma_2, L_1 \oplus L_2)\,,$$

where we have in view the identification $H^1(\Sigma_1 \sqcup \Sigma_2; \mathbb{R}) = H^1(\Sigma_1; \mathbb{R}) \oplus H^1(\Sigma_2; \mathbb{R})$.
(ii) If (h_1, m_1) and (h_2, m_2) are e-2-morphisms, then

$$(h_1, m_1) \sqcup (h_2, m_2) \;=\; (h_1 \sqcup h_2, m_1 + m_2)$$

(iii) If (X_1, L_1, n_1) and (X_2, L_2, n_2) are e-3-manifolds, their disjoint union is the e-3-manifold

$$(X_1, L_1, n_1) \sqcup (X_2, L_2, n_2) \;=\; (X_1 \sqcup X_2, L_1 \oplus L_2, n_1 + n_2)$$

Definition 6.8. *Gluing e-3-manifolds.*

Let $(\tilde{X}, \tilde{L}, \tilde{n})$ be an e-3-manifold such that $\partial(\tilde{X}, \tilde{L}, \tilde{n}) = (\partial\tilde{X}, \tilde{L}) = (Y, L) \sqcup (-\Sigma_1, L_1) \sqcup (\Sigma_2, L_2)$ and assume there exists an e-2-morphism $(h, m) : (\Sigma_1, L_1) \to (\Sigma_2, L_2)$. Under these assumptions, the gluing of $(\tilde{X}, \tilde{L}, \tilde{n})$ by (h, m) is defined to be the e-3-manifold

$$(\tilde{X}, \tilde{L}, \tilde{n})_{(h,m)} = \big(X, L, \tilde{n} + m + \tau(\tilde{L}, L_{\tilde{X}}, L_X \oplus L_h) \pmod 8\big)\,,$$

where X is the 3-manifold obtained by gluing $\tilde{X}$ by the diffeomorphism $h : \Sigma_1 \to \Sigma_2$ and where $\tilde{L}, L_{\tilde{X}}$ and $L_X \oplus L_h$ are the Lagrangian subspaces of $H^1(\partial\tilde{X}; \mathbb{R}) = H^1(Y; \mathbb{R}) \oplus H^1(-\Sigma_1; \mathbb{R}) \oplus H^1(\Sigma_2; \mathbb{R})$ defined as follows:

• $\tilde{L} = L \oplus L_1 \oplus L_2$ by the initial assumption on $(\tilde{X}, \tilde{L}, \tilde{n})$;

• $L_{\tilde{X}} = \operatorname{Im}\{H^1(\tilde{X}; \mathbb{R}) \longrightarrow H^1(\partial\tilde{X}; \mathbb{R})\}$;

• $L_X \oplus L_h$ is the direct sum of $L_X = \operatorname{Im}\{H^1(X; \mathbb{R}) \to H^1(Y; \mathbb{R})\} \subset H^1(Y; \mathbb{R})$ and of the Lagrangian subspace $L_h \subset H^1(-\Sigma_1; \mathbb{R}) \oplus H^1(\Sigma_2; \mathbb{R})$ defined as the graph of the symplectic isomorphism $h^* : H^1(\Sigma_2; \mathbb{R}) \to H^1(\Sigma_1; \mathbb{R})$, that is, $L_h = \{(h^*[\dot\eta], [\dot\eta]) \mid [\dot\eta] \in H^1(\Sigma_2; \mathbb{R})\}$.

Having introduced the necessary definitions, we are ready now to construct the full TQFT. For each e-2-manifold (Σ, L) we have a finite dimensional Hilbert space $\mathcal{H}(\Sigma, L)$ by the construction of Sect.5. Now let (X, L, n) be an e-3-manifold. Then to the e-2-manifold $(\partial X, L)$ obtained as the boundary of (X, L, n) there corresponds the Hilbert space $\mathcal{H}(\partial X, L)$. On the other hand, the 3-manifold X determines the Lagrangian subspace $L_X \subset H^1(\partial X; \mathbb{R})$ and therefore an e-2-manifold $(\partial X, L_X)$ with corresponding Hilbert space $\mathcal{H}(\partial X, L_X)$. Let

$$F_{LL_X} : \mathcal{H}(\partial X, L_X) \longrightarrow \mathcal{H}(\partial X, L) \tag{6.9}$$

be the unitary isomorphism induced by the BKS pairing (5.4). Then we define the vector $Z_{(X,L,n)}$ in $\mathcal{H}(\partial X, L)$ associated to the e-3-manifold (X, L, n) by

$$Z_{(X,L,n)} = \mathrm{e}^{\frac{\pi \mathrm{i}}{4} n} F_{LL_X}(Z_X)\,, \tag{6.10}$$

where Z_X is the standard vector in $\mathcal{H}(\partial X, L_X)$ defined by the expression (5.15).

The following theorem shows that we have a unitary TQFT.

Theorem 6.11. *The assignments*

$$\begin{aligned} \textit{e-2-manifold} \quad (\Sigma, L) &\longmapsto \textit{Hilbert space } \mathcal{H}(\Sigma, L) \\ \textit{e-3-manifold } (X, L, n) &\longmapsto \textit{vector } Z_{(X,L,n)} \in \mathcal{H}(\partial X, L) \end{aligned}$$

satisfy:

(a) Functoriality

To each e-2-morphism $(h, m) : (\Sigma', L') \to (\Sigma, L)$ *there corresponds a unitary isomorphism*

$$U(h, m) : \mathcal{H}(\Sigma, L) \longrightarrow \mathcal{H}(\Sigma', L')$$

and these compose properly.

Let $(\Phi, m) : (X', L', n') \longrightarrow (X, L, n)$ *be an e-3-morphism between isomorphic e-3-manifolds and* $(\partial\Phi, m) : (\partial X', L') \longrightarrow (\partial X, L)$ *the induced e-2-morphism between the boundaries. Then*

$$U(\partial\Phi, m)\, Z_{(X,L,n)} \;=\; Z_{(X',L',n')}$$

(b) Orientation

There is a natural isomorphism of Hilbert spaces

$$\mathcal{H}(-\Sigma, L) \cong \overline{\mathcal{H}(\Sigma, L)}$$

and

$$Z_{(-X,L,n)} = \overline{Z_{(X,L,n)}}$$

(c) Disjoint union

If $(\Sigma, L) = (\Sigma_1, L_1) \sqcup (\Sigma_2, L_2)$ *is a disjoint union of e-2-manifolds, then there is a natural unitary isomorphism*

$$\mathcal{H}(\Sigma_1 \sqcup \Sigma_2, L_1 \oplus L_2) \cong \mathcal{H}(\Sigma_1, L_1) \otimes \mathcal{H}(\Sigma_2, L_2)$$

If $(X, L, n) = (X_1, L_1, n_1) \sqcup (X_2, L_2, n_2)$ *is a disjoint union of e-3-manifolds, then*

$$Z_{(X_1 \sqcup X_2, L_1 \oplus L_2, n_1 + n_2)} \;=\; Z_{(X_1,L_1,n_1)} \otimes Z_{(X_2,L_2,n_2)}$$

(d) Cylinder axiom

If Σ is a closed oriented 2-manifold, $I = [0,1]$ the unit interval and $L_\Sigma \subset H^1(\Sigma;\mathbb{R})$ a rational Lagrangian subspace, then to the e-3-manifold $(\Sigma \times I, L_\Sigma \oplus L_\Sigma, 0)$ there corresponds

$$Z_{(\Sigma\times I, L_\Sigma \oplus L_\Sigma, 0)} \;=\; Id \,:\, \mathcal{H}(\Sigma, L_\Sigma) \longrightarrow \mathcal{H}(\Sigma, L_\Sigma)$$

(e) Gluing

Let X be a compact connected oriented 3-manifold with boundary ∂X and let X^{cut} denote the manifold obtained by cutting X along a codimension one closed oriented submanifold Σ. Then $\partial X^{cut} = \partial X \sqcup (-\Sigma) \sqcup \Sigma$. Let $Z_{(X,L,n)}$ be the vector associated to the e-3-manifold (X, L, n) and $Z_{(X^{cut}, L\oplus L_\Sigma \oplus L_\Sigma, n^{cut})}$ the vector for the cut e-3-manifold $(X^{cut}, L \oplus L_\Sigma \oplus L_\Sigma, n^{cut})$, with $L_\Sigma \subset H^1(\Sigma;\mathbb{R})$ an arbitrary rational Lagrangian subspace and

$$n^{cut} = n - \tau(L \oplus L_\Sigma \oplus L_\Sigma, L_{X^{cut}}, L_X \oplus L_\Delta) \pmod 8\,.$$

The Lagrangian subspace $L_\Delta \subset H^1(-\Sigma;\mathbb{R}) \oplus H^1(\Sigma;\mathbb{R})$ is the diagonal. Then to the gluing of e-3-manifolds

$$(X^{cut}, L \oplus L_\Sigma \oplus L_\Sigma, n^{cut})_{(id_\Sigma, 0)} \;=\; (X, L, n)$$

there corresponds the quantum gluing property

$$Z_{(X,L,n)} \;=\; \mathrm{Tr}_\Sigma\Big[Z_{(X^{cut}, L\oplus L_\Sigma\oplus L_\Sigma, n^{cut})}\Big]$$

where the operator Tr_Σ is the contraction

$$\mathrm{Tr}_\Sigma : \mathcal{H}(\partial X^{cut}, L \oplus L_\Sigma \oplus L_\Sigma) \cong \mathcal{H}(\partial X, L) \otimes \overline{\mathcal{H}(\Sigma, L_\Sigma)} \otimes \mathcal{H}(\Sigma, L_\Sigma) \to \mathcal{H}(\partial X, L)$$

using the hermitian inner product in $\mathcal{H}(\Sigma, L_\Sigma)$.

Proof. (a) *Functoriality.* The orientation preserving diffeomorphism $h : \Sigma' \to \Sigma$ induces a map in cohomology $h^* : H^1(\Sigma; *) \to H^1(\Sigma'; *)$ and therefore a symplectic diffeomorphism $h^* : \mathcal{M}_\Sigma \to \mathcal{M}_{\Sigma'}$ between the moduli spaces of flat $\mathbb{T}$-connections. Since there exist lifts of h to bundle morphisms between trivializable $\mathbb{T}$-bundles

over Σ' and trivializable $\mathbb{T}$-bundles over Σ, it follows from (4.4 (a)) and (4.16) that there is an induced isomorphism $h^* : \mathcal{L}_\Sigma \to \mathcal{L}_{\Sigma'}$ of prequantum line bundles covering $h^* : \mathcal{M}_\Sigma \to \mathcal{M}_{\Sigma'}$. The Lagrangian subspace $L \subset H^1(\Sigma;\mathbb{R})$ is mapped onto the Lagrangian subspace $h^*L \subset H^1(\Sigma';\mathbb{R})$. The map of line bundles

$$\begin{array}{ccc}
\mathcal{L}_\Sigma \otimes |\mathrm{Det}\mathcal{P}_L^*|^{\frac{1}{2}} & \xrightarrow{h^*} & \mathcal{L}_{\Sigma'} \otimes |\mathrm{Det}\mathcal{P}_{h^*L}^*|^{\frac{1}{2}} \\
\downarrow & & \downarrow \\
\mathcal{M}_\Sigma & \xrightarrow{h^*} & \mathcal{M}_{\Sigma'}
\end{array}$$

induces a map of sections which, having in view the constructions of Sect.5, gives rise to a unitary map of Hilbert spaces

$$h^* \,:\, \mathcal{H}(\Sigma, L) \longrightarrow \mathcal{H}(\Sigma', h^*L)\,.$$

The composition of this map with the isomorphism

$$F_{L',h^*L} \,:\, \mathcal{H}(\Sigma', h^*L) \longrightarrow \mathcal{H}(\Sigma', L')$$

arising from the BKS pairing between $\mathcal{H}(\Sigma', L')$ and $\mathcal{H}(\Sigma', h^*L)$ gives the unitary operator $F_{L',h^*L} \circ h^* \,:\, \mathcal{H}(\Sigma, L) \longrightarrow \mathcal{H}(\Sigma', L')$. Thus we are led to associate to the e-2-morphism (h, m) the unitary operator

$$U(h,m) \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}}{4} m}\, F_{L',h^*L} \circ h^* \,:\, \mathcal{H}(\Sigma, L) \longrightarrow \mathcal{H}(\Sigma', L')$$

We note that if L and K are rational Lagrangian subspaces in $H^1(\Sigma;\mathbb{R})$, then it follows from the definition of the intertwining isomorphisms (5.3) that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{H}(\Sigma, L) & \xrightarrow{F_{K,L}} & \mathcal{H}(\Sigma, K) \\
{\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h^*} \\
\mathcal{H}(\Sigma', h^*L) & \xrightarrow{F_{h^*K,h^*L}} & \mathcal{H}(\Sigma', h^*K)
\end{array}$$

Now let $(h', m') : (\Sigma'', L'') \to (\Sigma', L')$ be another e-2-morphism. Then, using the composition property (5.6), the above commutative diagram property and the

multiplication law of e-2-morphisms, we obtain

$$\begin{aligned} U(h',m')\,U(h,m) &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}(m+m')}\, F_{L'',h'^*L'} \circ h'^* \circ F_{L',h^*L} \circ h^* \\ &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}(m+m')}\, F_{L'',h'^*L'} \circ F_{h'^*L',h'^*h^*L} \circ h'^* \circ h^* \\ &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}[m+m'+\tau(L'',h'^*L',(hh')^*L)]}\, F_{L'',(hh')^*L} \circ (hh')^* \\ &= U((h,m)(h',m')) \end{aligned}$$

This proves the first statement in (6.11 (a)).

We prove now the second statement in (6.11 (a)). The orientation preserving diffeomorphism of compact oriented 3-manifolds $\Phi : X' \to X$ induces an isomorphism $\Phi^* : \mathcal{M}_X \to \mathcal{M}_{X'}$ between the moduli spaces of flat $\mathbb{T}$-connections. Its restriction $(\partial\Phi)^* : \mathcal{M}_{\partial X} \to \mathcal{M}_{\partial X'}$ lifts to an isomorphism of prequantum line bundles $(\partial\Phi)^* : \mathcal{L}_{\partial X} \to \mathcal{L}_{\partial X'}$. Moreover we have $r_{X'} \circ \Phi^* = (\partial\Phi)^* \circ r_X$, where $r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}$ and $r_{X'} : \mathcal{M}_{X'} \to \mathcal{M}_{\partial X'}$ are the restriction maps to the boundaries. Therefore $(\partial\Phi)^* L_X = L_{X'}$, so there is an induced bundle isomorphism $(\partial\Phi)^* : \mathcal{L}_{\partial X} \otimes |\mathrm{Det}\mathcal{P}_X^*|^{\frac{1}{2}} \to \mathcal{L}_{\partial X'} \otimes \left|\mathrm{Det}\mathcal{P}_{X'}^*\right|^{\frac{1}{2}}$. From the construction in Sect.5 of the standard vectors Z_X and $Z_{X'}$ as sections of these line bundles it follows that

$$(\partial\Phi)^*(Z_X) = Z_{X'}$$

under the unitary map of Hilbert spaces $(\partial\Phi)^* : \mathcal{H}(\partial X, L_X) \to \mathcal{H}(\partial X', L_{X'})$. On the other hand to the e-2-morphism $(\partial\Phi, m) : (\partial X', L') \to (\partial X, L)$ there corresponds the unitary operator

$$U(\partial\Phi, m) = \mathrm{e}^{\frac{\pi \mathrm{i}}{4} m}\, F_{L',(\partial\Phi)^*L} \circ (\partial\Phi)^* : \mathcal{H}(\partial X, L) \to \mathcal{H}(\partial X', L')$$

According to the definition (5.15) of vectors associated to e-3-manifolds we have $Z_{(X,L,n)} = \mathrm{e}^{\frac{\pi \mathrm{i}}{4} n} F_{LL_X}(Z_X)$ and $Z_{(X',L',n')} = \mathrm{e}^{\frac{\pi \mathrm{i}}{4} n'} F_{L'L_{X'}}(Z_{X'})$. Thus we find

$$\begin{aligned} U(\partial\Phi, m) Z_{(X,L,n)} &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}(n+m)} F_{L',(\partial\Phi)^* L} \circ (\partial\Phi)^* \circ F_{LL_X}(Z_X) \\ &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}(n+m)} F_{L',(\partial\Phi)^* L} \circ F_{(\partial\Phi)^* L, (\partial\Phi)^* L_X} \circ (\partial\Phi)^*(Z_X) \\ &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}[n+m+\tau((\partial\Phi)^* L_X, L', (\partial\Phi)^* L)]} F_{L',(\partial\Phi)^* L_X}(Z_{X'}) \\ &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}[n+m+\tau(L_{X'}, L', (\partial\Phi)^* L)]} F_{L' L_{X'}}(Z_{X'}) \\ &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4} n'} F_{L' L_{X'}}(Z_{X'}) \;=\; Z_{(X',L',n')} \end{aligned}$$

This concludes the proof of (6.11(a)).

(b) *Orientation.* Follows from (4.4 (b)) and (4.18 (b)).

(c) *Disjoint union.* To the disjoint union $\Sigma = \Sigma_1 \sqcup \Sigma_2$ there corresponds the product of symplectic spaces

$$(\mathcal{M}_\Sigma, \omega_\Sigma) = (\mathcal{M}_{\Sigma_1}, \omega_{\Sigma_1}) \times (\mathcal{M}_{\Sigma_2}, \omega_{\Sigma_2}) = (\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}, pr^*_{\Sigma_1}\omega_{\Sigma_1} + pr^*_{\Sigma_2}\omega_{\Sigma_2})$$

where $pr_{\Sigma_i} : \Sigma_1 \times \Sigma_2 \to \Sigma_i$ are the natural projections. If $L_i \subset H^1(\Sigma_i; \mathbb{R})$ are rational Lagrangian subspaces, then $L_1 \oplus L_2 \subset H^1(\Sigma_1 \sqcup \Sigma_2; \mathbb{R}) = H^1(\Sigma_1; \mathbb{R}) \oplus H^1(\Sigma_2; \mathbb{R})$ is again a rational Lagrangian subspace. The Bohr-Sommerfeld orbits Λ of the polarization $P_{L_1 \oplus L_2}$ on $\mathcal{M}_{\Sigma_1} \times \mathcal{M}_{\Sigma_2}$ are Cartesian products $\Lambda_1 \times \Lambda_2$ of Bohr-Sommerfeld orbits Λ_i for the polarizations $\mathcal{P}_{L_i}$ on $\mathcal{M}_{\Sigma_i}$, $i = 1, 2$. The proof of (c) follows then from (4.4 (c)) and (4.18 (c)) and the definitions of the Hilbert spaces associated to e-2-manifolds and of the vectors associated to e-3-manifolds.

(d) *Cylinder axiom.* Since $H^i(\Sigma \times I) \cong H^i(\Sigma)$, $i = 0, 1, 2$, we obtain in particular that $\mathcal{M}_{\Sigma \times I} \cong \mathcal{M}_\Sigma$. As $\partial(\Sigma \times I) = (-\Sigma) \sqcup \Sigma$ we have $\mathcal{M}_{\partial(\Sigma \times I)} = \mathcal{M}_\Sigma \times \mathcal{M}_\Sigma$ and the restriction map

$$r_{\Sigma \times I} : \mathcal{M}_{\Sigma \times I} \cong \mathcal{M}_\Sigma \longrightarrow \mathcal{M}_{\partial(\Sigma \times I)} = \mathcal{M}_\Sigma \times \mathcal{M}_\Sigma$$

is the diagonal map $[\eta] \mapsto ([\eta], [\eta])$. Hence the Lagrangian subspace $L_{\Sigma \times I} = L_\Delta$, with L_Δ denoting the diagonal in $H^1(-\Sigma; \mathbb{R}) \oplus H^1(\Sigma; \mathbb{R})$ and the Lagrangian submanifold $\Lambda_{\Sigma \times I} = \Delta$, with Δ the diagonal in $\mathcal{M}_\Sigma \times \mathcal{M}_\Sigma$. According to the definitions and constructions of Sect.5 we find that the Chern-Simons section over

Δ of the prequantum line bundle is $\sigma_{\Sigma\times I} = 1$ and that the section $\mu_{\Sigma\times I}$ over Δ of the half-density bundle $|\mathrm{Det}\,\mathcal{P}^*_\Delta|^{\frac{1}{2}}$ is identified under the isomorphism $\Delta \cong \mathcal{M}_\Sigma$ with the square root $(T_\Sigma)^{\frac{1}{2}}$ of the R-torsion of Σ. In [Wi2] it is proved that the density T_Σ on $\mathcal{M}_\Sigma$ coincides with the density $\left|\frac{(\omega_\Sigma)^g}{g!}\right|$ determined by the symplectic form ω_Σ, where $g = \frac{1}{2}\dim H^1(\Sigma;\mathbb{R})$. Having in view the definition (5.15), we find therefore that the standard vector in $\mathcal{H}(-\Sigma \sqcup \Sigma, L_\Delta)$ associated to the 3-manifold $\Sigma\times I$ is

$$Z_{\Sigma\times I} \;=\; k^{\frac{1}{4}\dim H^1(\Sigma;\mathbb{R})}\,\sigma_{\Sigma\times I}\otimes\mu_{\Sigma\times I} \;=\; k^{\frac{1}{4}\dim H^1(\Sigma;\mathbb{R})}\,1\otimes(T_\Sigma)^{\frac{1}{2}}\,. \tag{6.12}$$

Let us consider now the e-3-manifold $(\Sigma\times I, L_{2\Sigma}, n)$, where $L_{2\Sigma} = L_\Sigma \oplus L_\Sigma$ with $L_\Sigma \subset H^1(\Sigma;\mathbb{R})$ an arbitrary rational Lagrangian subspace. To this e-3-manifold there corresponds the vector in $\mathcal{H}(-\Sigma\sqcup\Sigma, L_{2\Sigma}) \cong \overline{\mathcal{H}(\Sigma, L_\Sigma)} \otimes \mathcal{H}(\Sigma, L_\Sigma)$ defined by

$$Z_{(\Sigma\times I, L_{2\Sigma}, n)} \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}}{4}n}\,F_{L_{2\Sigma},L_\Delta}(Z_{\Sigma\times I})\,.$$

Let us introduce a unitary basis $\{v(\Sigma,L_\Sigma)_{\mathbf{q}}\}$ for the Hilbert space $\mathcal{H}(\Sigma, L_\Sigma)$. Using the definition of the unitary operator $F_{L_{2\Sigma},L_\Delta} : \mathcal{H}(-\Sigma\sqcup\Sigma, L_\Delta) \longrightarrow \mathcal{H}(-\Sigma\sqcup\Sigma, L_{2\Sigma})$ in terms of the BKS pairing (5.4), we find that

$$Z_{(\Sigma\times I, L_{2\Sigma}, n)} \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}}{4}n}\sum_{\mathbf{q}} \overline{v(\Sigma,L_\Sigma)_{\mathbf{q}}} \otimes v(\Sigma,L_\Sigma)_{\mathbf{q}}\,. \tag{6.13}$$

Thus $Z_{(\Sigma\times I, L_{2\Sigma}, 0)} = Id \in \mathrm{Hom}[\mathcal{H}(\Sigma, L_\Sigma), \mathcal{H}(\Sigma, L_\Sigma)]$.

(e) *Gluing.* The moduli space of flat $\mathbb{T}$-connections on $\partial X^{cut} = \partial X \sqcup (-\Sigma) \sqcup \Sigma$ is the symplectic manifold $(\mathcal{M}_{\partial X^{cut}}, \omega_{\partial X^{cut}})$ with

$$\begin{aligned}
\mathcal{M}_{\partial X^{cut}} \;&=\; \mathcal{M}_{\partial X}\times\mathcal{M}_\Sigma\times\mathcal{M}_\Sigma \\
\omega_{\partial X^{cut}} \;&=\; pr^*_{\partial X}(\omega_{\partial X}) \,+\, pr^*_{-\Sigma}(-\omega_\Sigma) \,+\, pr^*_\Sigma(\omega_\Sigma)\,,
\end{aligned}$$

where $pr_{\partial X}, pr_{-\Sigma}$ and pr_Σ denote the projection maps from $\mathcal{M}_{\partial X}\times\mathcal{M}_\Sigma\times\mathcal{M}_\Sigma$ onto the first, second and third factor, respectively. Let $\Delta \subset \mathcal{M}_\Sigma\times\mathcal{M}_\Sigma$ denote the diagonal. We claim that the manifold

$$C \;=\; \mathcal{M}_{\partial X}\times\Delta \;\subset\; \mathcal{M}_{\partial X^{cut}}$$

is a coisotropic submanifold of $(\mathcal{M}_{\partial X^{cut}}, \omega_{\partial X^{cut}})$. Let $TC^{\perp}$ denote the orthogonal complement of the tangent bundle TC with respect to $\omega_{\partial X^{cut}}$, i.e.

$$TC^{\perp} = \{v \in T\mathcal{M}_{\partial X^{cut}} \mid \omega_{\partial X^{cut}}(v, w) = 0, \text{ for all } w \in TC\}.$$

For any vector $v = (v_{\partial X}, v_{\Sigma}, v'_{\Sigma}) \in T\mathcal{M}_{\partial X^{cut}}$ such that $v \in TC^{\perp}$ we must have $0 = \omega_{\partial X^{cut}}(v, w) = \omega_{\partial X}(v_{\partial X}, w_{\partial X}) - \omega_{\Sigma}(v_{\Sigma}, w_{\Sigma}) + \omega_{\Sigma}(v'_{\Sigma}, w_{\Sigma})$, for all vectors $w = (w_{\partial X}, w_{\Sigma}, w_{\Sigma}) \in TC$. This is true if and only if $v_{\partial X} = 0$ and $v_{\Sigma} = v'_{\Sigma}$. Thus $TC^{\perp} \subset TC$ and $C \subset \mathcal{M}_{\partial X^{cut}}$ is coisotropic. Moreover, $TC^{\perp}$ is the tangent bundle to an isotropic foliation $C^{\perp}$ of C. The reduced symplectic manifold $C/C^{\perp}$ is identified with $(\mathcal{M}_{\partial X}, \omega_{\partial X})$ and the quotient map $C \to C/C^{\perp}$ with the natural projection $pr_{\partial X} : \mathcal{M}_{\partial X} \times \Delta \to \mathcal{M}_{\partial X}$.

With the usual notations let $\Lambda_{X^{cut}}$ be the image of $\mathcal{M}_{X^{cut}}$ under the restriction map $r_{X^{cut}} : \mathcal{M}_{X^{cut}} \to \mathcal{M}_{\partial X^{cut}}$. According to (2.5), $\Lambda_{X^{cut}}$ is a Lagrangian submanifold of $(\mathcal{M}_{\partial X^{cut}}, \omega_{\partial X^{cut}})$ and we consider its reduction [We] relative to the coisotropic submanifold C. The intersection $\Lambda_{X^{cut}} \cap C$ is a manifold and the tangent bundles satisfy $T(\Lambda_{X^{cut}} \cap C) = T\Lambda_{X^{cut}} \cap TC$. The reduction of $\Lambda_{X^{cut}}$ relative to C is the image of $\Lambda_{X^{cut}} \cap C$ under the quotient map $C \to C/C^{\perp}$. It is a Lagrangian submanifold of $C/C^{\perp}$ which coincides with $\Lambda_X \subset \mathcal{M}_{\partial X}$ under the identification of $C/C^{\perp}$ with $\mathcal{M}_{\partial X}$. That is,

$$\Lambda_X = pr_{\partial X}(\Lambda_{X^{cut}} \cap C).$$

The kernel of the differential of $p_{\partial X} : \Lambda_{X^{cut}} \cap C \to \Lambda_X$ is the vector bundle $T\Lambda_{X^{cut}} \cap TC^{\perp}$.

Now let $g : X^{cut} \to X$ denote the gluing map and $g^* : \mathcal{M}_X \to \mathcal{M}_{X^{cut}}$ the induced map between the moduli spaces of flat $\mathbb{T}$-connections. If $r_X : \mathcal{M}_X \to \mathcal{M}_{\partial X}$ and $r_{\Sigma} : \mathcal{M}_X \to \mathcal{M}_{\Sigma}$ are the restriction maps from connections on X to connections on ∂X and Σ, respectively, then we get the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_X & \xrightarrow{r_X \times r_{\Sigma}} & \mathcal{M}_{\partial X} \times \mathcal{M}_{\Sigma} \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle id \times i_{\Delta}} \\ \mathcal{M}_{X^{cut}} & \xrightarrow{r_{X^{cut}}} & \mathcal{M}_{\partial X^{cut}} = \mathcal{M}_{\partial X} \times \mathcal{M}_{\Sigma} \times \mathcal{M}_{\Sigma} \end{array} \tag{6.14}$$

where $i_\Delta : \mathcal{M}_\Sigma \to \mathcal{M}_\Sigma \times \mathcal{M}_\Sigma$ is the diagonal map $i_\Delta([\eta]) = ([\eta],[\eta]) \in \Delta$. We note that

$$\Lambda_{X^{cut}} \cap C \;=\; (r_{X^{cut}} \circ g^*)(\mathcal{M}_X)\,. \tag{6.15}$$

Recall that $\mathcal{M}_{X^{cut}} = \bigsqcup_{p^{cut}\in \mathrm{Tors} H^2(X^{cut};\mathbb{Z})} \mathcal{M}_{X^{cut},p^{cut}}$ and $\mathcal{M}_X = \bigsqcup_{p\in \mathrm{Tors} H^2(X;\mathbb{Z})} \mathcal{M}_{X,p}$. Thus for each p^{cut} and each connected component $\big[\Lambda_{X^{cut}} \cap C\big]_c$ of $\Lambda_{X^{cut}} \cap C$ there is a unique p such that

$$r^{-1}_{X^{cut}}\big(\big[\Lambda_{X^{cut}} \cap C\big]_c\big) \cap \mathcal{M}_{X^{cut},p^{cut}} \;=\; g^*(\mathcal{M}_{X,p})\,. \tag{6.16}$$

Then (6.16) and (2.2 (ii)) imply the following relation:

$$\big[\#\pi_0(\Lambda_{X^{cut}} \cap C)\big]\cdot\big[\#\mathrm{Tors}\, H^2(X^{cut};\mathbb{Z})\big] \;=\; \big[\#\mathrm{Tors} H^2(X;\mathbb{Z})\big] \tag{6.17}$$

With the usual notations let

$$\begin{aligned} L_{X^{cut}} \;&=\; \mathrm{Im}\,\{H^1(X^{cut};\mathbb{R}) \to H^1(\partial X^{cut};\mathbb{R})\} \\ L_X \;&=\; \mathrm{Im}\,\{H^1(X;\mathbb{R}) \to H^1(\partial X;\mathbb{R})\} \end{aligned}$$

and let us introduce the following Lagrangian subspaces of the symplectic vector space $H^1(X^{cut};\mathbb{R}) = H^1(X;\mathbb{R}) \oplus H^1(-\Sigma;\mathbb{R}) \oplus H^1(\Sigma;\mathbb{R})$:

$$\begin{aligned} L' \;&=\; L_X \oplus L_\Sigma \oplus L_\Sigma \\ L'' \;&=\; L \oplus L_\Sigma \oplus L_\Sigma\,. \end{aligned} \tag{6.18}$$

The vector corresponding to the e-3-manifold (X^{cut}, L'', n^{cut}) is the vector

$$Z_{(X^{cut},L'',n^{cut})} \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}}{4} n^{cut}}\, F_{L'' L_{X^{cut}}}(Z_{X^{cut}}) \;\in\; \mathcal{H}(\partial X^{cut}, L'')\,, \tag{6.19}$$

obtained as the image of the standard vector

$$Z_{X^{cut}} \;=\; \frac{k^{m_{X^{cut}}}}{[\#\mathrm{Tors}\, H^2(X^{cut};\mathbb{Z})]}\, \sigma_{X^{cut}} \otimes \mu_{X^{cut}} \;\in\; \mathcal{H}(\partial X^{cut}, L_{X^{cut}}) \tag{6.20}$$

defined in Sect.5. Using the composition law

$$F_{L'' L_{X^{cut}}} \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\tau(L_{X^{cut}},L',L'')}\, F_{L''L'} \circ F_{L' L_{X^{cut}}}$$

we can rewrite (6.19) as

$$Z_{(X^{cut},L'',n^{cut})} \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}}{4}[n^{cut}+\tau(L_{X^{cut}},L',L'')]}\, F_{L''L'} \circ F_{L'L_{X^{cut}}}(Z_{X^{cut}}) \tag{6.21}$$

Let us choose unitary bases $\{v(\partial X,L_X)_{\mathbf{q}}\}$ for $\mathcal{H}(\partial X, L_X)$, $\{v(\partial X,L)_{\mathbf{q}'}\}$ for $\mathcal{H}(\partial X, L)$ and $\{v(\Sigma,L_\Sigma)_{\boldsymbol{\ell}}\}$ for $\mathcal{H}(\Sigma, L_\Sigma)$. Then we have

(6.22)

$$\begin{aligned} F_{L''L'} \circ F_{L'L_{X^{cut}}}(Z_{X^{cut}}) \;&=\; F_{L''L'}\Big[\sum_{\mathbf{q},\boldsymbol{\ell},\boldsymbol{\ell}'} M_{\mathbf{q}\boldsymbol{\ell}\boldsymbol{\ell}'}\, v(\partial X,L_X)_{\mathbf{q}} \otimes \overline{v(\Sigma,L_\Sigma)_{\boldsymbol{\ell}}} \otimes v(\Sigma,L_\Sigma)_{\boldsymbol{\ell}'}\Big] \\ &=\; \sum_{\mathbf{q},\boldsymbol{\ell},\boldsymbol{\ell}'} M_{\mathbf{q}\boldsymbol{\ell}\boldsymbol{\ell}'}\,\Big[F_{LL_X}\, v(\partial X,L_X)_{\mathbf{q}}\Big] \otimes \overline{v(\Sigma,L_\Sigma)_{\boldsymbol{\ell}}} \otimes v(\Sigma,L_\Sigma)_{\boldsymbol{\ell}'}\,, \end{aligned}$$

where $F_{LL_X} : \mathcal{H}(\partial X, L_X) \to \mathcal{H}(\partial X, L)$ and

$$M_{\mathbf{q}\boldsymbol{\ell}\boldsymbol{\ell}'} \;=\; \Big\langle v(\partial X,L_X)_{\mathbf{q}} \otimes \overline{v(\Sigma,L_\Sigma)_{\boldsymbol{\ell}}} \otimes v(\Sigma,L_\Sigma)_{\boldsymbol{\ell}'}\,,\; F_{L'L_{X^{cut}}}(Z_{X^{cut}})\Big\rangle_{\mathcal{H}(\partial X^{cut},L')}\,.$$

The last equality in (6.22) follows from the definition (5.5) of the intertwining isomorphism $F_{L''L'}$ through the BKS pairing (5.4) which, having in view the special form (6.18) of the Lagrangian subspaces L' and L'', leads to the expression $F_{L''L'} = F_{LL_X} \otimes Id \otimes Id$.

Applying the contraction operator Tr_Σ to $Z_{(X^{cut},L'',n^{cut})}$ and making use of the expression (6.22), we get

$$\mathrm{Tr}_\Sigma\Big[Z_{(X^{cut},L'',n^{cut})}\Big] \;=\; \mathrm{e}^{\frac{\pi \mathrm{i}}{4}[n^{cut}+\tau(L_{X^{cut}},L',L'')]} \sum_{\mathbf{q},\boldsymbol{\ell}} M_{\mathbf{q}\boldsymbol{\ell}\boldsymbol{\ell}}\, F_{LL_X}\Big[v(\partial X,L_X)_{\mathbf{q}}\Big]\,.$$

Now let $L_\Delta \subset H^1(-\Sigma;\mathbb{R}) \oplus H^1(\Sigma;\mathbb{R})$ be the diagonal. It is obviously a rational Lagrangian subspace for the symplectic form $pr^*_{-\Sigma}(-\omega_\Sigma) + pr^*_\Sigma(\omega_\Sigma)$ on $H^1(-\Sigma;\mathbb{R}) \oplus H^1(\Sigma;\mathbb{R})$. We introduce the notation $L_{2\Sigma} = L_\Sigma \oplus L_\Sigma$ and define $L'_\Delta = L_X \oplus L_\Delta$. Recall that to the cylinder $\Sigma \times I$ there corresponds the standard vector $Z_{\Sigma\times I}$ in $\mathcal{H}((-\Sigma)\sqcup\Sigma, L_\Delta)$ given by the expression (6.12). Associated to the e-3-manifold $(\Sigma \times I, L_{2\Sigma}, 0)$ we have, according to (6.13), the vector

$$Z_{(\Sigma\times I,L_{2\Sigma},0)} \;=\; F_{L_{2\Sigma},L_\Delta}(Z_{\Sigma\times I}) \;=\; \sum_{\boldsymbol{\ell}} \overline{v(\Sigma,L_\Sigma)_{\boldsymbol{\ell}}} \otimes v(\Sigma,L_\Sigma)_{\boldsymbol{\ell}}$$

in $\mathcal{H}((-\Sigma)\sqcup\Sigma, L_{2\Sigma})$. Using this we can write the following:

$$\begin{aligned}
\sum_{\boldsymbol{\ell}} M_{\mathbf{q}\boldsymbol{\ell}\boldsymbol{\ell}} &= \sum_{\boldsymbol{\ell}} \Big\langle \upsilon(\partial X, L_X)_{\mathbf{q}} \otimes \overline{\upsilon(\Sigma, L_\Sigma)_{\boldsymbol{\ell}}} \otimes \upsilon(\Sigma, L_\Sigma)_{\boldsymbol{\ell}} \, , \, F_{L' L_{X^{cut}}}(Z_{X^{cut}}) \Big\rangle_{\mathcal{H}(\partial X^{cut}, L')} \\
&= \Big\langle \upsilon(\partial X, L_X)_{\mathbf{q}} \otimes Z_{(\Sigma\times I, L_{2\Sigma}, 0)} \, , \, F_{L' L_{X^{cut}}}(Z_{X^{cut}}) \Big\rangle_{\mathcal{H}(\partial X^{cut}, L')} \\
&= \Big\langle \upsilon(\partial X, L_X)_{\mathbf{q}} \otimes F_{L_{2\Sigma}, L_\Delta}(Z_{\Sigma\times I}) \, , \, F_{L' L_{X^{cut}}}(Z_{X^{cut}}) \Big\rangle_{\mathcal{H}(\partial X^{cut}, L')}
\end{aligned}$$

Now, since $F_{L' L_{X^{cut}}} = \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\tau(L_{X^{cut}}, L'_\Delta, L')}\, F_{L' L'_\Delta} \circ F_{L'_\Delta, L_{X^{cut}}}$ and since the operator $F_{L' L'_\Delta} = Id \otimes F_{L_{2\Sigma}, L_\Delta}$, we obtain

$$\begin{aligned}
\sum_{\boldsymbol{\ell}} M_{\mathbf{q}\boldsymbol{\ell}\boldsymbol{\ell}} &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\tau(L_{X^{cut}}, L'_\Delta, L')} \; \times \\
&\times \Big\langle F_{L' L'_\Delta}\big[\upsilon(\partial X, L_X)_{\mathbf{q}} \otimes Z_{\Sigma\times I}\big] \, , \, F_{L' L'_\Delta} \circ F_{L'_\Delta, L_{X^{cut}}}(Z_{X^{cut}}) \Big\rangle_{\mathcal{H}(\partial X^{cut}, L')} \\
&= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\tau(L_{X^{cut}}, L'_\Delta, L')} \Big\langle \upsilon(\partial X, L_X)_{\mathbf{q}} \otimes Z_{\Sigma\times I} \, , \, F_{L'_\Delta, L_{X^{cut}}}(Z_{X^{cut}}) \Big\rangle_{\mathcal{H}(\partial X^{cut}, L'_\Delta)} \\
&= \frac{k^{m_{X^{cut}} + \frac{1}{4}\dim H^1(\Sigma;\mathbb{R})}}{\big[\#\mathrm{Tors}\, H^2(X^{cut};\mathbb{Z})\big]} \, \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\tau(L_{X^{cut}}, L'_\Delta, L')} \; \times \\
&\times \int_{\Lambda_{X^{cut}} \cap ((\Lambda_X)_{\mathbf{q}} \times \Delta)} \big(s(\partial X)_{\mathbf{q}} \otimes \sigma_{\Sigma\times I} \, , \, \sigma_{X^{cut}}\big) \, \big(\delta_X \otimes \mu_{\Sigma\times I}\big) * \mu_{X^{cut}}
\end{aligned}$$

The last equality follows from the BKS pairing formula defining the operator $F_{L'_\Delta, L_{X^{cut}}}$ and the expressions (6.20) for $Z_{X^{cut}}$ and (6.12) for $Z_{\Sigma\times I}$. We also wrote $\upsilon(\partial X, L_X)_{\mathbf{q}} = s(\partial X)_{\mathbf{q}} \otimes \delta_X$ where δ_X is the invariant $\frac{1}{2}$-density on $\mathcal{P}_X$ such that $\int_{(\Lambda_X)_{\mathbf{q}}} \delta_X^2 = 1$, for any leaf $(\Lambda_X)_{\mathbf{q}}$ of the polarization $\mathcal{P}_X$ on $\mathcal{M}_{\partial X}$.

As previously shown we have the reduction map $\Lambda_{X^{cut}} \cap C \to \Lambda_X$ and, since $\Lambda_X \cap (\Lambda_X)_{\mathbf{q}} \neq \emptyset$ if and only if $(\Lambda_X)_{\mathbf{q}} = \Lambda_X$, we conclude that

$$\Lambda_{X^{cut}} \cap ((\Lambda_X)_{\mathbf{q}} \times \Delta) = \emptyset, \quad \text{for all leaves } (\Lambda_X)_{\mathbf{q}} \neq \Lambda_X \, .$$

Let us write $\mu_{X^{cut}} = a_{X^{cut}} \delta_{X^{cut}}$ with $\delta_{X^{cut}}$ the invariant $\frac{1}{2}$-density on $\mathcal{P}_{X^{cut}}$ satisfying $\int_{\Lambda_{X^{cut}}} \delta_{X^{cut}}^2 = 1$. We also recall from (6.12) that $\mu_{\Sigma\times I} = (T_\Sigma)^{\frac{1}{2}} = \left|\frac{(\omega_\Sigma)^g}{g!}\right|^{\frac{1}{2}}$ and that we have $\int_\Delta \frac{(\omega_\Sigma)^g}{g!} = 1$, where $g = \frac{1}{2}\dim H^1(\Sigma;\mathbb{R})$. Then, making use of

the results in ([Ma],§4), we find that for each connected component $\left[\Lambda_{X^{cut}} \cap C\right]_c$ of the manifold $\Lambda_{X^{cut}} \cap C = \Lambda_{X^{cut}} \cap (\Lambda_X \times \Delta)$ we have

$$\int\limits_{\left[\Lambda_{X^{cut}} \cap C\right]_c} \left(\delta_X \otimes T_\Sigma^{\frac{1}{2}}\right) * \delta_{X^{cut}} = k^{\frac{1}{2}[\dim(\Lambda_{X^{cut}} \cap C) - \frac{1}{2}\dim H^1(\partial X^{cut};\mathbb{R})]} \times \times \left[\#\pi_0(\Lambda_{X^{cut}} \cap C)\right]^{-\frac{1}{2}}$$

Let us introduce the notation $v(\partial X, L_X) = v(\partial X, L_X)_{\mathbf{q}}$ if $(\Lambda_X)_{\mathbf{q}} = \Lambda_X$. We also note that the function $\left(s(\partial X) \otimes \sigma_{\Sigma\times I}, \sigma_{\partial X^{cut}}\right)$ on $\Lambda_{\partial X^{cut}} \cap C$ is constant on each connected component. Therefore, summarizing the previous results and observations and using the fact that $\sigma_{X^{cut}} = \sum\limits_{p^{cut} \in \mathrm{Tors} H^2(X^{cut};\mathbb{Z})} \sigma_{X^{cut},p^{cut}}$, we can write

(6.23)
$$\begin{aligned}
\mathrm{Tr}_\Sigma\left[Z_{(X^{cut},L'',n^{cut})}\right] &= \mathrm{e}^{\frac{\pi \mathrm{i}}{4}[n^{cut} + \tau(L_{X^{cut}},L',L'') + \tau(L_{X^{cut}},L'_\Delta,L')]} \\
&\times \frac{a_{X^{cut}}}{\left[\#\mathrm{Tors}H^2(X^{cut};\mathbb{Z})\right]} \left[\#\pi_0(\Lambda_{X^{cut}} \cap C)\right]^{-\frac{1}{2}} \\
&\times k^{m_{X^{cut}} + \frac{1}{4}\dim H^1(\Sigma;\mathbb{R}) + \frac{1}{2}\dim(\Lambda_{X^{cut}} \cap C) - \frac{1}{4}\dim H^1(\partial X^{cut};\mathbb{R})} \\
&\times \Big[\sum_c \sum_{p^{cut} \in \mathrm{Tors}H^2(X^{cut};\mathbb{Z})} \left(s(\partial X) \otimes \sigma_{\Sigma\times I}, \sigma_{X^{cut},p^{cut}}\right)_c\Big] \\
&\times F_{LL_X}\left(v(\partial X, L_X)\right),
\end{aligned}$$

where $\left(s(\partial X) \otimes \sigma_{\Sigma\times I}, \sigma_{X^{cut},p^{cut}}\right)_c$ denotes the value of the function on the component $[\Lambda_{X^{cut}} \cap C]_c$. Then let $[\Theta] \in \mathcal{M}_{X,p}$ with $[\eta] = [\Theta\big|_\Sigma]$ and $[\partial\Theta] = [\Theta\big|_{\partial X}]$ and let $[\Theta^{cut}] = [g^*\Theta]$. Using the classical gluing formula (4.19) and the definitions of $\sigma_{\Sigma\times I}$, $\sigma_{X,p}$ and $\sigma_{X^{cut},p^{cut}}$, we find that

$$\begin{aligned}
&\Big(\sigma_{X,p}([\partial\Theta]) \otimes \sigma_{\Sigma\times I}([\eta] \sqcup [\eta])\,,\, \sigma_{X^{cut},p^{cut}}([\partial\Theta] \sqcup [\eta] \sqcup [\eta])\Big) \\
&\qquad = \left(\mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)}\,,\, \mathrm{Tr}_\eta\left[\mathrm{e}^{\pi \mathrm{i} k S_{X^{cut},P^{cut}}(\Theta^{cut})}\right]\right) = 1\,.
\end{aligned}$$

Since both $\sigma_{X,p}$ and $s(\partial X)$ are unitary and covariantly constant sections of $\mathcal{L}_{\partial X} \otimes |\mathrm{Det}\mathcal{P}_X^*|^{\frac{1}{2}}$ over Λ_X, they differ by a constant phase factor. In view of this observation, of the previous relation and of the relations (6.16)-(6.17), the expression

(6.23) becomes

(6.24)
$$\begin{aligned}\mathrm{Tr}_\Sigma\Big[Z_{(X^{cut},L'',n^{cut})}\Big] \;&=\; \mathrm{e}^{\frac{\pi \mathrm{i}}{4}[n^{cut}+\tau(L_{X^{cut}},L',L'')+\tau(L_{X^{cut}},L'_\Delta,L')]} \\ &\times\; \frac{a_{X^{cut}}}{\big[\#\mathrm{Tors}H^2(X^{cut};\mathbb{Z})\big]^{\frac12}\big[\#\mathrm{Tors}H^2(X;\mathbb{Z})\big]^{\frac12}} \\ &\times\; k^{m_{X^{cut}}+\frac14\dim H^1(\Sigma;\mathbb{R})+\frac12\dim(\Lambda_{X^{cut}}\cap C)-\frac14\dim H^1(\partial X^{cut};\mathbb{R})} \\ &\times\; \sum_{p\in\mathrm{Tors}H^2(X;\mathbb{Z})} F_{LL_X}(\sigma_{X,p}\otimes\delta_X)\,.\end{aligned}$$

Let a_X denote the real constant such that $\mu_X = a_X\delta_X$. We claim that the following relations hold:

$$a^2_{X^{cut}} \;=\; a^2_X\,\frac{\big[\#\mathrm{Tors}H^2(X^{cut};\mathbb{Z})\big]}{\big[\#\mathrm{Tors}H^2(X;\mathbb{Z})\big]} \tag{6.25}$$

and

(6.26)
$$m_{X^{cut}}+\frac14\dim H^1(\Sigma;\mathbb{R})+\frac12\dim(\Lambda_{X^{cut}}\cap C)-\frac14\dim H^1(\partial X^{cut};\mathbb{R}) \;=\; m_X\,.$$

We postpone for a moment their proof and note first that by using the definition of n^{cut} and the cocycle relation [LV] for the Maslov-Kashiwara index τ we have:

$$\begin{aligned}&n^{cut}+\tau(L_{X^{cut}},L',L'')+\tau(L_{X^{cut}},L'_\Delta,L') \pmod 8\\ &=\; n-\tau(L'',L_{X^{cut}},L'_\Delta)+\tau(L_{X^{cut}},L',L'')+\tau(L_{X^{cut}},L'_\Delta,L') \pmod 8\\ &=\; n+\tau(L',L'',L'_\Delta) \pmod 8.\end{aligned} \tag{6.27}$$

From the symplectic additivity property of τ we find

$$\begin{aligned}\tau(L',L'',L'_\Delta) \;&=\; \tau(L_X\oplus L_\Sigma\oplus L_\Sigma, L\oplus L_\Sigma\oplus L_\Sigma, L_X\oplus L_\Delta)\\ &=\; \tau(L_X,L,L_X)+\tau(L_\Sigma\oplus L_\Sigma, L_\Sigma\oplus L_\Sigma, L_\Delta)=0\,.\end{aligned} \tag{6.28}$$

Putting the results (6.24)-(6.28) together we obtain

$$\begin{aligned}\mathrm{Tr}_\Sigma\Big[Z_{(X^{cut},L'',n^{cut})}\Big] \;&=\; \mathrm{e}^{\frac{\pi\mathrm{i}}{4}n}\,\frac{k^{m_X}}{\big[\#\mathrm{Tors}H^2(X;\mathbb{Z})\big]}\,F_{LL_X}(\sigma_X\otimes\mu_X)\\ &=\; \mathrm{e}^{\frac{\pi\mathrm{i}}{4}n}\,F_{LL_X}(Z_X) \;=\; Z_{(X,L,n)}\end{aligned}$$

which proves the gluing property (6.11 (e)).

Let us return now to the proof of the equality (6.26). Recall that m_X is defined by (5.16) and similarly $m_{X^{cut}}$ is given by

$$\begin{aligned} m_{X^{cut}} \;=\; &\frac{1}{4}\big(\dim H^1(X^{cut};\mathbb{R}) + \dim H^1(X^{cut},\partial X^{cut};\mathbb{R}) \\ &- \dim H^0(X^{cut};\mathbb{R}) - \dim H^0(X^{cut},\partial X^{cut};\mathbb{R})\big)\,. \end{aligned}$$

We have the following relations:

(i) The Mayer-Vietoris cohomology sequence for the spaces X, X^{cut}, Σ

$$\cdots \to H^i(X;\mathbb{R}) \to H^i(X^{cut};\mathbb{R}) \to H^i(\Sigma;\mathbb{R}) \to H^{i+1}(X;\mathbb{R}) \to \cdots \tag{6.29}$$

together with Poincaré duality imply the relation

$$\begin{aligned} &\dim H^1(X^{cut};\mathbb{R}) - \dim H^1(X^{cut},\partial X^{cut};\mathbb{R}) \\ &\qquad - \dim H^0(X^{cut};\mathbb{R}) + \dim H^0(X^{cut},\partial X^{cut};\mathbb{R}) \\ &\qquad = \dim H^1(X;\mathbb{R}) - \dim H^1(X,\partial X;\mathbb{R}) - \dim H^0(X;\mathbb{R}) \\ &\qquad + \dim H^0(X,\partial X;\mathbb{R}) + \dim H^1(\Sigma;\mathbb{R}) - 2\dim H^0(\Sigma;\mathbb{R})\,. \end{aligned} \tag{6.30}$$

(ii) The exact cohomology sequence for the pair of spaces $(\partial X \sqcup \Sigma) \subset X$ and (6.14)-(6.15) give

$$\begin{aligned} &0 \to H^0(X;\mathbb{R}) \to H^0(\partial X \sqcup \Sigma;\mathbb{R}) \to H^1(X,\partial X \sqcup \Sigma;\mathbb{R}) \to \\ &\qquad \to H^1(X;\mathbb{R}) \xrightarrow{\dot r_X \times (\dot r_\Sigma, \dot r_\Sigma)} T(\Lambda_{X^{cut}} \cap C) \to 0 \end{aligned}$$

Together with the isomorphism $H^1(X,\partial X \sqcup \Sigma;\mathbb{R}) \cong H^1(X^{cut},\partial X^{cut};\mathbb{R})$ it implies that

$$\begin{aligned} \dim(\Lambda_{X^{cut}} \cap C) \;=\; &\dim H^1(X;\mathbb{R}) - \dim H^1(X^{cut},\partial X^{cut};\mathbb{R}) \\ &+ \dim H^0(\partial X;\mathbb{R}) + \dim H^0(\Sigma;\mathbb{R}) - \dim H^0(X;\mathbb{R})\,. \end{aligned} \tag{6.31}$$

(iii) From the exact sequence (2.8) and the fact that $\dim \Lambda_X = \frac{1}{2}\dim H^1(\partial X;\mathbb{R})$ we obtain

$$\begin{aligned} \frac{1}{2}\dim H^1(\partial X;\mathbb{R}) \;=\; &\dim H^1(X;\mathbb{R}) - \dim H^1(X,\partial X;\mathbb{R}) \\ &+ \dim H^0(\partial X;\mathbb{R}) - \dim H^0(X;\mathbb{R}) \\ &+ \dim H^0(X,\partial X;\mathbb{R}) \end{aligned} \tag{6.32}$$

The claimed formula (6.26) follows then from the relations (6.30), (6.31) and (6.32).

Finally let us prove the relation (6.25). Referring back to the definition (5.13) we have

$$\begin{aligned} \mu_X &= \left[\int\limits_{H^1(X,\partial X;\mathbb{T})} \mathbf{w} \right] (T_X)^{\frac{1}{2}} \otimes w^{-1} \\ \mu_{X^{cut}} &= \left[\int\limits_{H^1(X^{cut},\partial X^{cut};\mathbb{T})} \mathbf{w}_{cut} \right] (T_{X^{cut}})^{\frac{1}{2}} \otimes w_{cut}^{-1}\,, \end{aligned} \tag{6.33}$$

with $w \in \left|\mathrm{Det} H^1(X,\partial X;\mathbb{R})^*\right|$ and $w_{cut} \in \left|\mathrm{Det} H^1(X^{cut},\partial X^{cut};\mathbb{R})^*\right|$. On the other hand we introduced a_X and $a_{X^{cut}}$ such that

$$\begin{aligned} \mu_X &= a_X \delta_X \\ \mu_{X^{cut}} &= a_{X^{cut}} \delta_{X^{cut}} \end{aligned} \tag{6.34}$$

The R-torsion T_X of X is related to the R-torsions $T_{X^{cut}}$ of X^{cut} and T_Σ of Σ by the gluing formula [V]

$$T_X = T_{X^{cut}} \otimes (T_\Sigma)^{-1}\,. \tag{6.35}$$

The identification between the l.h.s. and the r.h.s. is made through the isomorphism of determinant lines

$$\left|\mathrm{Det} H^\bullet(X;\mathbb{R})^*\right| \cong \left|\mathrm{Det} H^\bullet(X^{cut};\mathbb{R})^*\right| \otimes \left|\mathrm{Det} H^\bullet(\Sigma;\mathbb{R})\right|$$

arising from the Mayer-Vietoris sequence (6.29). Using the exact sequence (2.8) and, due to the normalization of the density δ_X^2 on Λ_X, we see that

$$\frac{w \otimes \delta_X^2}{\left[\int\limits_{H^1(X,\partial X;\mathbb{T})} \mathbf{w} \right]} \in \left|\mathrm{Det} H^1(X;\mathbb{R})^*\right|$$

defines an invariant density on the group $H^1(X;\mathbb{T})$, which gives $H^1(X;\mathbb{T})$ volume 1. Similarly on $H^1(X^{cut};\mathbb{T})$ we have the density defined by

$$\frac{w_{cut} \otimes \delta_{X^{cut}}^2}{\left[\int\limits_{H^1(X^{cut},\partial X^{cut};\mathbb{T})} \mathbf{w}_{cut} \right]} \in \left|\mathrm{Det} H^1(X^{cut};\mathbb{R})^*\right|$$

The Mayer-Vietoris sequence (6.29) and Poincaré duality induce the isomorphism of determinant lines

$$\left|\mathrm{Det} H^1(X;\mathbb{R})^*\right| \cong \left|\mathrm{Det} H^1(X^{cut};\mathbb{R})^*\right| \otimes \left|\mathrm{Det} H^1(X^{cut},\partial X^{cut};\mathbb{R})^*\right|$$
$$\otimes \left|\mathrm{Det} H^1(\Sigma;\mathbb{R})\right| \otimes \left|\mathrm{Det} H^1(X,\partial X;\mathbb{R})\right|$$

Thus we can define an invariant density ρ_X on the group $H^1(X;\mathbb{T})$ by setting

$$\rho_X = \frac{w_{cut} \otimes \delta^2_{X^{cut}}}{\left[\int\limits_{H^1(X^{cut},\partial X^{cut};\mathbb{T})} \mathbf{w}_{cut}\right]} \otimes \frac{w_{cut}}{\left[\int\limits_{H^1(X^{cut},\partial X^{cut};\mathbb{T})} \mathbf{w}_{cut}\right]} \otimes T_\Sigma^{-1} \otimes \frac{w^{-1}}{\left[\int\limits_{H^1(X,\partial X;\mathbb{T})} \mathbf{w}\right]^{-1}}$$

From the normalization of the densities on the r.h.s. and from the Mayer-Vietoris cohomology sequence with $\mathbb{T}$-coefficients for X, X^{cut}, Σ

$$\cdots \to H^1(X;\mathbb{T}) \to H^1(X^{cut};\mathbb{T}) \to H^1(\Sigma;\mathbb{T})$$
$$\to H^2(X;\mathbb{T}) \to H^2(X^{cut};\mathbb{T}) \to H^2(\Sigma;\mathbb{T}) \to 0$$

we see that, since $\pi_0[H^2(X;\mathbb{T})] \cong \pi_0[H^2(X^{cut};\mathbb{T})] = 0$ and since, as shown in Prop. (2.2), $\pi_0[H^1(X,\partial X;\mathbb{T})] \cong \mathrm{Tors} H^2(X,\partial X;\mathbb{Z}) \cong \mathrm{Tors} H^2(X;\mathbb{Z})$ and similarly $\pi_0[H^1(X^{cut},\partial X^{cut};\mathbb{T})] \cong \mathrm{Tors} H^2(X^{cut},\partial X^{cut};\mathbb{Z}) \cong \mathrm{Tors} H^2(X^{cut};\mathbb{Z})$, the density ρ_X gives $H^1(X;\mathbb{T})$ volume equal to $\frac{\left[\#\mathrm{Tors} H^2(X;\mathbb{Z})\right]}{\left[\#\mathrm{Tors} H^2(X^{cut};\mathbb{Z})\right]}$. Making use of the gluing

formula (6.35) and the relations (6.33) and (6.34) we can write

$$\begin{aligned}
\frac{w\otimes\delta_X^2}{\left[\int\limits_{H^1(X,\partial X;\mathbb{T})}\mathbf{w}\right]} &= \frac{1}{a_X^2}\,T_X\otimes\frac{w^{-1}}{\left[\int\limits_{H^1(X,\partial X;\mathbb{T})}\mathbf{w}\right]^{-1}}\\
&= \frac{1}{a_X^2}\,T_{X^{cut}}\otimes(T_\Sigma)^{-1}\otimes\frac{w^{-1}}{\left[\int\limits_{H^1(X,\partial X;\mathbb{T})}\mathbf{w}\right]^{-1}}\\
&= \frac{a_{X^{cut}}^2}{a_X^2}\,\frac{w_{cut}^2\otimes\delta_{X^{cut}}^2}{\left[\int\limits_{H^1(X^{cut},\partial X^{cut};\mathbb{T})}\mathbf{w}_{cut}\right]^2}\otimes(T_\Sigma)^{-1}\otimes\frac{w^{-1}}{\left[\int\limits_{H^1(X,\partial X;\mathbb{T})}\mathbf{w}\right]^{-1}}\\
&= \frac{a_{X^{cut}}^2}{a_X^2}\,\rho_X
\end{aligned}$$

In view of the previous observation regarding the volume of $H^1(X;\mathbb{T})$ computed with ρ_X and the fact that the density of the above equation gives $H^1(X;\mathbb{T})$ volume 1, the claimed relation (6.25) is proved. $\square$

The mapping class group Γ_Σ of a closed oriented 2-manifold Σ is the group $\mathrm{Diff}_+(\Sigma)/\mathrm{Diff}_0(\Sigma)$ of isotopy classes of orientation preserving diffeomorphisms of Σ. The mapping class group $\Gamma_{(\Sigma,L)}$ of an e-2-manifold (Σ,L) is a central extension by $\mathbb{Z}/8\mathbb{Z}$ of Γ_Σ:

$$1\longrightarrow\mathbb{Z}/8\mathbb{Z}\longrightarrow\Gamma_{(\Sigma,L)}\longrightarrow\Gamma_\Sigma\longrightarrow 1$$

As a set $\Gamma_{(\Sigma,L)}=\Gamma_\Sigma\times\mathbb{Z}/8\mathbb{Z}$ and the composition law in $\Gamma_{(\Sigma,L)}$ is, according to (6.4), given by

$$([h],m)([h'],m')=\big([hh'],\,m+m'+\tau(L'',h'^*L',(hh')^*L)\pmod 8\big)$$

In the course of proving Theorem (6.11 (a)) we showed that, given a rational Lagrangian subspace $L\subset H^1(\Sigma;\mathbb{R})$, each element $[h]$ of the mapping class group Γ_Σ determines a unitary map of Hilbert spaces $h^*:\mathcal{H}(\Sigma,L)\to\mathcal{H}(\Sigma,h^*L)$. The composition of this map with the isomorphism $F_{L,h^*L}:\mathcal{H}(\Sigma,h^*L)\to\mathcal{H}(\Sigma,L)$

induced by the BKS pairing is a unitary operator $U_L([h]) = F_{L,h^*L} \circ h^*$ on the Hilbert space $\mathcal{H}(\Sigma, L)$. The assignment

$$[h] \mapsto U_L([h])$$

defines a unitary projective representation of Γ_Σ on $\mathcal{H}(\Sigma, L)$. For the group $\Gamma_{(\Sigma,L)}$, the assignment

$$([h], m) \mapsto U_L([h], m) = \mathrm{e}^{\frac{\pi \mathrm{i}}{4} m} U_L([h])$$

determines a unitary representation of this group on the Hilbert space $\mathcal{H}(\Sigma, L)$.

Since the mapping class group Γ_Σ acts on $H^1(\Sigma; \mathbb{Z})$, there is a natural homomorphism from Γ_Σ to the group $Sp(\mathcal{Z})$ of symplectic transformations of $(H^1(\Sigma; \mathbb{R}), \omega_\Sigma)$ which preserve the integer lattice $\mathcal{Z} = H^1(\Sigma; \mathbb{Z})$. According to the results in ([Ma],§9.1), for each rational Lagrangian subspace $L \subset H^1(\Sigma; \mathbb{R})$, there is a projective unitary representation of $Sp(\mathcal{Z})$ on the Hilbert space $\mathcal{H}(\Sigma, L)$. Let us choose an integer symplectic basis $(\mathbf{w}; \mathbf{w}')$ for $H^1(\Sigma; \mathbb{R})$, with $w_1, \ldots, w_g$ spanning L, where $g = \frac{1}{2} \dim H^1(\Sigma; \mathbb{R})$. As shown in ([Ma],§3), the choice of such a basis uniquely determines a unitary basis $\{v(\Sigma,L)_{\mathbf{q}}\}_{\mathbf{q} \in (\mathbb{Z}/k\mathbb{Z})^g}$ for $\mathcal{H}(\Sigma, L)$. The generators of $Sp(\mathcal{Z})$ are the elements with matrix form

$$\alpha = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \quad \beta = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \quad \gamma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

with respect to the basis $(\mathbf{w}; \mathbf{w}')$, where $A \in GL(g, \mathbb{Z})$ and $B \in M(g, \mathbb{Z})$, ${}^tB = B$. Then the projective unitary representation of $Sp(\mathcal{Z})$ on $\mathcal{H}(\Sigma, L)$ is described by the following operators representing the generators:

$$\begin{aligned} U_L(\alpha)\, v(\Sigma,L)_{\mathbf{q}} &= v(\Sigma,L)_{{}^tA^{-1}\mathbf{q}} \\ U_L(\beta)\, v(\Sigma,L)_{\mathbf{q}} &= \mathrm{e}^{\frac{\pi \mathrm{i}}{k}\, {}^t\mathbf{q} B \mathbf{q}}\, v(\Sigma,L)_{\mathbf{q}} \\ U_L(\gamma)\, v(\Sigma,L)_{\mathbf{q}} &= k^{-\frac{g}{2}} \sum_{\mathbf{q_1} \in (\mathbb{Z}/k\mathbb{Z})^g} \mathrm{e}^{\frac{2\pi \mathrm{i}}{k}\, {}^t\mathbf{q}\mathbf{q_1}}\, v(\Sigma,L)_{\mathbf{q_1}} \end{aligned} \tag{6.36}$$

For $g = 1$ we have the torus mapping class group $SL(2, \mathbb{Z})$ with standard generators the matrices $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, subject to the relations $(ST)^3 = I$, $S^4 = I$.

The projective unitary representation of $SL(2,\mathbb{Z})$ on the Hilbert space $\mathcal{H}(T^2, L)$ associated to the torus T^2 and $L \subset H^1(T^2;\mathbb{R})$ is described by the operators:

$$\begin{aligned} U_L(T)_{qq'} &= \mathrm{e}^{\frac{\pi \mathrm{i}}{k} q^2}\, \delta_{qq'} \\ U_L(S)_{qq'} &= k^{-\frac{1}{2}}\, \mathrm{e}^{\frac{2\pi \mathrm{i}}{k} qq'} \end{aligned}$$

It coincides with the representation of the modular group $\mathbb{P}SL(2,\mathbb{Z})$ in two dimensional rational conformal field theory [Ve]. The projective representation (6.36) of $Sp(\mathcal{Z})$ on $\mathcal{H}(\Sigma, L)$ is the same projective representation as the one constructed in [G] on the vector space of theta functions at level k obtained through the quantization of the moduli space of flat $\mathbb{T}$-connections on Σ in a holomorphic polarization.

7. The path integral approach

This section attempts to provide a motivation for the definition given at the end of Sect.5 of the object Z_X that we associate to a compact oriented 3-manifold X. Following Witten's approach [Wi1] we start by defining Z_X, for X a closed oriented 3-manifold, as a partition function, that is, a Feynman type functional integral (path integral) over gauge equivalence classes of $\mathbb{T}$-connections on X with action the Chern-Simons functional. For the definition of the path integral and its subsequent evaluation we rely almost entirely on the results in [Sch1, Sch2]. A similar derivation is presented in [A2] in the context of the non-abelian version of the Chern-Simons theory. In the abelian Chern-Simons theory the path integral is of Gaussian type and the so-called semi-classical or stationary phase approximation gives the exact result. For the case of a 3-manifold X with boundary, the path integral is defined over a space of connections which are fixed over the boundary. Hence it is a function of the boundary connections and leads, presumably, to an element in the Hilbert space associated to the boundary ∂X.

Remark 7.1. Let us begin with a brief description of a finite dimensional model which will serve as a prototype for the problem in infinite dimensions of defining

Z_X as a partition function. The main references used are [Sch1, Sch2], to which we refer the reader for more details and proofs.

Let G be a compact Lie group acting as a group of isometries of the Riemannian manifold (E, g_E). We assume that the Lie algebra $\operatorname{Lie} G$ of G is endowed with an invariant inner product which defines an invariant Riemannian metric on G. Then $\operatorname{Vol} G$ denotes the volume of G with respect to this metric. The action of G on E generates a homomorphism of $\operatorname{Lie} G$ into the Lie algebra of vector fields on E. This defines for each point $x \in E$ a linear map $\tau_x : \operatorname{Lie} G \to T_x E$. Let H_x denote the isotropy subgroup of G at the point $x \in E$ and $\operatorname{Vol} H_x$ the volume of H_x with respect to the metric induced by the invariant metric in G. We assume that the isotropy subgroups at all points $x \in E$ are conjugate to a fixed subgroup $H \subset G$. Then $\operatorname{Vol} H_x = \operatorname{Vol} H$, for all $x \in E$. The G-invariant Riemannian metric g_E on E induces a Riemannian metric $g_{E/G}$ on the space E/G of orbits of G in E. It is defined by setting for any $v, w \in T_{[x]}(E/G) \;:\; g_{E/G}(v, w) = g_E(\hat{v}, \hat{w})$, where $\hat{v}, \hat{w} \in T_x E$ project onto v, w and belong to the orthogonal complement of $\tau_x(\operatorname{Lie} G)$ in $T_x E$. We let μ_E and $\mu_{E/G}$ be the measures on E and E/G determined by these metrics. Then, if h is a G-invariant function on E, we have

$$\int_E h(x)\, \mu_E = \frac{\operatorname{Vol} G}{\operatorname{Vol} H} \int_{E/G} h(x) \left[\det{}'(\tau_x^* \tau_x)\right]^{\frac{1}{2}} \mu_{E/G}\, . \tag{7.2}$$

The notation $\det'$ refers to the regularized determinant of the operator, that is, the product of all the nonzero eigenvalues. The G-invariant term $\frac{\operatorname{Vol} G}{\operatorname{Vol} H}\left[\det{}'(\tau_x^* \tau_x)\right]^{\frac{1}{2}}$ is the volume of the orbit through the point x. A proof of the above formula is given in [Sch1].

Now let us consider a G-invariant real valued function f on E such that the stationary points of f form a G-invariant submanifold F of E and $f(x) = A$ for all $x \in F$. The Hessian $(\operatorname{Hess} f)_x$ of f at the point $x \in E$ is a linear self-adjoint operator $(\operatorname{Hess} f)_x : T_x E \to T_x E$ and, since the function f is G-invariant, $\tau_x(\operatorname{Lie} G) \subset \operatorname{Ker}(\operatorname{Hess} f)_x$. At every point x belonging to the critical manifold F,

the symmetric bilinear form on T_xE defined by

$$(\mathrm{Hess}f)_x(v,w) \;=\; g_E((\mathrm{Hess}f)_x v, w) \;=\; g_E(v, (\mathrm{Hess}f)_x w)\,, \tag{7.3}$$

has the expression $(\mathrm{Hess}f)_x(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \;=\; \frac{\partial^2 f}{\partial x^i \partial x^j}$ in local coordinates around x. We assume that $T_xF = \mathrm{Ker}(\mathrm{Hess}f)_x$ at every point $x \in F$. We also assume that the space $M = F/G$ of orbits of G in F is a manifold and we let μ_M denote the measure on M determined by the natural metric on M induced from the Riemannian metric of F. The method of stationary phase ([GS, Sch1]) together with the formula (7.2) give for the asymptotic evaluation of the integral

$$\frac{1}{\mathrm{Vol}\,G} \int\limits_E \mathrm{e}^{\pi \mathrm{i} a f(x)}\, \mu_E \qquad (a \in \mathbb{R}_+)\,, \tag{7.4}$$

in the limit of large a, the following expression [Sch1, Sch2]:

$$\frac{1}{a^{(\dim E - \dim F)/2}} \frac{\mathrm{e}^{\pi \mathrm{i} a A}}{\mathrm{Vol}\,H} \int\limits_M \mathrm{e}^{\frac{\pi \mathrm{i}}{4} \mathrm{sgn}(\mathrm{Hess}f)_x}\, \frac{\left|\det{}'(\tau_x^* \tau_x)\right|^{\frac{1}{2}}}{\left|\det{}'(\mathrm{Hess}f)_x\right|^{\frac{1}{2}}}\, \mu_M\,. \tag{7.5}$$

$\mathrm{sgn}(\mathrm{Hess}f)_x$ denotes the signature of the quadratic form in (7.3). In the terminology of [Sch1, Sch2], the integral in (7.4) is the *partition function* of the function(al) f.

In trying to apply the above result to the problem of defining Z_X as a functional integral one needs to make sense of determinants and signatures of operators analogous to the ones in (7.5), but this time in an infinite dimensional setting. This is accomplished through the use of zeta-regularization of determinants [RS1, Sch2] and regularization of signatures via eta-invariants [APS, A2]. We recall the relevant definitions.

Remark 7.6. *Zeta-regularization of determinants.* We follow [Sch2]. A non-negative self-adjoint operator B on a Hilbert space $\mathcal{H}$ is called regular if $\mathrm{Tr}[\mathrm{e}^{-tB} - \Pi(B)] = \sum_k \alpha_k(B) t^k$ as $t \to +0$, where k runs over a finite set of non-negative

numbers. $\Pi(B)$ denotes the projection operator onto the kernel of B. The zeta-function $\zeta_B(s)$ of a regular operator B is defined for large $\mathrm{Re}(s)$ by the expression

$$\zeta_B(s) \,=\, \sum_{\lambda_j \neq 0} \lambda_j^{-s} \,=\, \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}[\mathrm{e}^{-tB} - \Pi(B)]\, t^{s-1}\, dt\,,$$

where λ_j are the eigenvalues of B. The function $\zeta_B(s)$ admits a meromorphic continuation to $\mathbb{C}$ and is analytic at $s = 0$. The regularized determinant $\det' B$ is defined by the expression

$$\det{}' B \,=\, \mathrm{e}^{-\zeta_B'(0)} \tag{7.7}$$

For an operator $B : \mathcal{H}_0 \to \mathcal{H}_1$ between the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$, with adjoint operator B^* and such that B^*B is regular, the regularized determinant $\det' B$ is defined by

$$\det{}' B \,=\, \mathrm{e}^{-\frac{1}{2}\zeta_{B^*B}'(0)} \,=\, \big[\det{}'(B^*B)\big]^{\frac{1}{2}} \tag{7.8}$$

Remark 7.9. Let S be a self-adjoint operator an a Hilbert space $\mathcal{H}_1$. Let $T : \mathcal{H}_0 \to \mathcal{H}_1$ be an operator between the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ with adjoint $T^* : \mathcal{H}_1 \to \mathcal{H}_0$ and such that $T(\mathcal{H}_0) \subset \mathrm{Ker} S$. The operators S^2 and T^*T are assumed regular. By definition $\det' S = (\det' S^2)^{\frac{1}{2}}$ and $\det' T = (\det' T^*T)^{\frac{1}{2}}$. Then we have [Sch2]:

$$\det{}' S \,=\, \frac{\big[\det{}'(S^2 + TT^*)\big]^{\frac{1}{2}}}{\big[\det{}'(TT^*)\big]^{\frac{1}{2}}} \,=\, \frac{\big[\det{}'(S^2 + TT^*)\big]^{\frac{1}{2}}}{\big[\det{}'(T^*T)\big]^{\frac{1}{2}}} \tag{7.10}$$

Remark 7.11. *Regularization of signatures.* Let B be a self-adjoint operator on a Hilbert space and define the function

$$\eta_B(s) \,=\, \sum_{\lambda_j \neq 0} (\mathrm{sign}\lambda_j)|\lambda_j|^{-s} \tag{7.12}$$

with λ_j the eigenvalues of B. The function $\eta_B(s)$ can be meromorphically continued to $\mathbb{C}$ and has a removable singularity at $s = 0$. The *eta-invariant* of the operator B is defined as $\eta(B) = \eta_B(0)$ [APS]. If B is a finite dimensional matrix,

then $\eta_B(0)$ is the signature of B, i.e. the number of positive eigenvalues minus the number of negative eigenvalues.

We return now to the problem of defining Z_X as a functional integral. We consider first a *closed* connected oriented 3-manifold X. For each $p \in \mathrm{Tors} H^2(X;\mathbb{Z})$, we choose a $\mathbb{T}$-bundle P on X with $c_1(P) = p$. The space $\mathcal{A}_P$ of connections on P is an affine space with vector space $2\pi\mathrm{i}\Omega^1(X;\mathbb{R})$. The group of gauge transformations $\mathcal{G}_P \cong \mathcal{G}_X$ acts on $\mathcal{A}_P$ by (2.1) and, according to (3.7), the Chern-Simons functional $S_{X,P} : \mathcal{A}_P \to \mathbb{R}/\mathbb{Z}$ defined in (3.2) is invariant under this action. We define Z_X by the expression

$$Z_X = \sum_{p \in \mathrm{Tors} H^2(X;\mathbb{Z})} Z_{X,p}\,, \tag{7.13}$$

where $Z_{X,p}$ is the partition function of the Chern-Simons functional, that is,

$$Z_{X,p} = \int_{\mathcal{A}_P/\mathcal{G}_P} [\mathcal{D}\Theta]\, \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)}\,. \tag{7.14}$$

The functional integral on the r.h.s. of the above equation is just a formal expression meant to suggest that, pending the existence of a measure $[\mathcal{D}\Theta]$, we integrate over the space of gauge equivalence classes of connections on P. We are going to show in the following that this functional integral can be given a precise meaning.

Let us choose a Riemannian metric on X. For each $q = 0, 1, \ldots, \dim X$, the metric determines an inner product on the space $\Omega^q(X;\mathbb{R})$ of q-forms on X:

$$(\alpha, \beta) = \int_X \alpha \wedge \star\beta \tag{7.15}$$

In particular, it defines an inner product on the tangent space $T\mathcal{A}_P \cong 2\pi\mathrm{i}\Omega^1(X;\mathbb{R})$. This makes $\mathcal{A}_P$ a Riemannian manifold and the group $\mathcal{G}_P$ acts on $\mathcal{A}_P$ by isometries. We let $\hat{\mu}$ denote the measure on $\mathcal{A}_P$ determined by this metric and μ the induced measure on the quotient space $\mathcal{A}_P/\mathcal{G}_P$. For each $\Theta \in \mathcal{A}_P$, let τ_Θ denote the differential of the map from $\mathcal{G}_P$ to $\mathcal{A}_P$ which defines by (2.1) the $\mathcal{G}_P$-action. The linear map τ_Θ sends $\mathrm{Lie}\,\mathcal{G}_P \cong 2\pi\mathrm{i}\Omega^0(X;\mathbb{R})$ into $T_\Theta\mathcal{A}_P \cong 2\pi\mathrm{i}\Omega^1(X;\mathbb{R})$. Thus τ_Θ is identified with the exterior differential $\tau_\Theta = d : \Omega^0(X;\mathbb{R}) \to \Omega^1(X;\mathbb{R})$. The inner

product in Lie $\mathcal{G}_P \cong 2\pi i\Omega^0(X;\mathbb{R})$ defined by (7.15) induces an invariant metric on the group $\mathcal{G}_P$. The isotropy subgroup of $\mathcal{G}_P$ at a point $\Theta \in \mathcal{A}_P$ is the group of constant maps from X into $\mathbb{T}$; hence, it is isomorphic to $\mathbb{T}$. We let Vol $\mathbb{T}$ denote the volume of the isotropy subgroup, computed with respect to the metric induced from that of $\mathcal{G}_P$. Its evaluation gives

$$\operatorname{Vol}\mathbb{T} \;=\; \big[\operatorname{Vol} X\big]^{\frac{1}{2}} \;=\; \Big[\int_X \star 1\Big]^{\frac{1}{2}} \tag{7.16}$$

Thus, besides the fact that we are dealing with infinite dimensional spaces, the setting is the same as the one in Remark (7.1). We have the identifications $E = \mathcal{A}_P$, $G = \mathcal{G}_P$ and $f = kS_{X,P}$. According to (3.8) the stationary points of the Chern-Simons functional $S_{X,P}$ are the flat connections; hence the critical manifold F is the subspace $\mathcal{A}_P^f = \{\Theta \in \mathcal{A}_P \mid F_\Theta = d\Theta = 0\}$ and the constant value of the functional $S_{X,P}$ on $\mathcal{A}_P^f$ is the Chern-Simons invariant of flat connections on P. If Θ_P is a flat connection in $\mathcal{A}_P$, any $\Theta \in \mathcal{A}_P$ can be written as $\Theta = \Theta_P + 2\pi i A$, for some $A \in \Omega^1(X;\mathbb{R})$. Then, since $\partial X = \emptyset$ and $F_{\Theta_P} = 0$, we obtain from (3.18) that

$$\begin{aligned} S_{X,P}(\Theta) \;&=\; S_{X,P}(\Theta_P) + \int_X \langle 2\pi i A \wedge d(2\pi i A)\rangle \pmod 1 \\ &=\; S_{X,P}(\Theta_P) - (A, \star dA) \pmod 1 \end{aligned}$$

Thus the Hessian of the functional $f = kS_{X,P}$ is the linear operator Hess$f = -k\star d : \Omega^1(X;\mathbb{R}) \to \Omega^1(X;\mathbb{R})$. The quotient manifold $\mathcal{M}_P = \mathcal{A}_P^f/\mathcal{G}_P$ is isomorphic to the torus $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$. The inner product (7.15) on 1-forms determines a natural inner product on $H^1(X;\mathbb{R})$ through the Hodge-deRham isomorphism $H^1(X;\mathbb{R}) \cong \mathcal{H}^1(X)$ with the space of harmonic 1-forms. Since the tangent space $T_{[\Theta]}\mathcal{M}_P \cong H^1(X;\mathbb{R})$, at any $[\Theta] \in \mathcal{M}_P$, this inner product defines a measure ν on $\mathcal{M}_P$.

In view of these observations and of the results (7.2) and (7.5) from the model of Remark (7.1), we *formally* write:

$$Z_{X,p} = \frac{1}{\mathrm{Vol}\,\mathcal{G}_P} \int\limits_{\mathcal{A}_P} \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)}\,\hat{\mu} \tag{7.17}$$

$$= \frac{1}{\mathrm{Vol}\,\mathbb{T}} \int\limits_{\mathcal{A}_P/\mathcal{G}_P} \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta)} \big[\det{}'(\tau_\Theta^* \tau_\Theta)\big]^{\frac{1}{2}}\,\mu \tag{7.17 $'$}$$

$$= \frac{\mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta_P)}}{\mathrm{Vol}\,\mathbb{T}} \int\limits_{\mathcal{M}_P} \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\mathrm{sgn}(-\star d)} \frac{\big[\det{}'(\tau_\Theta^* \tau_\Theta)\big]^{\frac{1}{2}}}{\big[\det{}'(-k\star d)\big]^{\frac{1}{2}}}\,\nu \tag{7.17 $''$}$$

The equality (7.17$''$) is the result of the fact that, since $S_{X,P}$ is a quadratic functional, the stationary phase method which gives for an oscillatory integral of the type (7.4) the asymptotic evaluation (7.5), produces in this case the exact result. The expression (7.17$'$) will be taken as the formal definition for $Z_{X,p}$ in (7.14). Although the r.h.s. of (7.17) is meaningless, the expression in (7.17$''$) has rigorous mathematical meaning if the determinants and signatures of the operators therein are regularized according to the definitions in Remark (7.6) and (7.11). The signature of the operator $-\star d$ on $\Omega^1(X;\mathbb{R})$ is regularized via the eta-invariant, that is we take $\mathrm{sgn}(-\star d) = \eta(-\star d)$. If $\Delta_q = d^*d + dd^*$ is the Laplacian on $\Omega^q(X;\mathbb{R})$ and if we let $\zeta_q = \zeta_{\Delta_q}$, then for any real number $\lambda > 0$

$$\det{}'(\lambda \Delta_q) = \lambda^{\zeta_q(0)} \det{}'(\Delta_q) \tag{7.18}$$

and [Mü]

$$\zeta_q(0) = -\dim \mathrm{Ker}\Delta_q = -\dim H^q(X;\mathbb{R}) \tag{7.19}$$

Now, let us make use of the results stated in remark (7.9) to evaluate the regularized determinant $\det{}'(-k\star d)$. Thus, we let $S = -k\star d$ acting on $\Omega^1(X;\mathbb{R})$ and $T = kd : \Omega^0(X;\mathbb{R}) \to \Omega^1(X;\mathbb{R})$. Then $S^2 + TT^* = k^2(d^*d + dd^*) = k^2\Delta_1$ and $T^*T = k^2 d^*d = k^2\Delta_0$. Using (7.10) and (7.18)-(7.19) we get

$$\det{}'(-k\star d) = \frac{\big[\det{}'(k^2\Delta_1)\big]^{\frac{1}{2}}}{\big[\det{}'(k^2\Delta_0)\big]^{\frac{1}{2}}} = \frac{k^{-\dim H^1(X;\mathbb{R})}}{k^{-\dim H^0(X;\mathbb{R})}} \frac{\big[\det{}'\Delta_1\big]^{\frac{1}{2}}}{\big[\det{}'\Delta_0\big]^{\frac{1}{2}}} \tag{7.20}$$

We also note that $\tau_\Theta^*\tau_\Theta = d^*d = \Delta_0$. Inserting the above results into the expression (7.17″) we obtain

$$Z_{X,p} \;=\; k^{m_X}\, \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta_P)} \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\eta(-\star d)} \int\limits_{\mathcal{M}_P} \frac{1}{\big[\mathrm{Vol} X\big]^{\frac{1}{2}}} \frac{\big[\det{}'\,\Delta_0\big]^{\frac{3}{4}}}{\big[\det{}'\,\Delta_1\big]^{\frac{1}{4}}}\,\nu \tag{7.21}$$

where $m_X = \frac{1}{2}\big(\dim H^1(X;\mathbb{R}) - \dim H^0(X;\mathbb{R})\big)$. The expression under the integral sign in the above formula is related to the Ray-Singer analytic torsion of X.

The analytic torsion of a closed Riemannian manifold was introduced in [RS1, RS2] as a norm on the determinant line of the deRham cohomology of the manifold. Thus, for the closed 3-manifold X endowed with a Riemannian metric, the Ray-Singer analytic torsion is a density

$$\begin{aligned} T^a_X \in |\mathrm{Det} H^\bullet(X;\mathbb{R})^*| = |\mathrm{Det} H^0(X;\mathbb{R})| &\otimes |\mathrm{Det} H^1(X;\mathbb{R})^*| \\ &\otimes |\mathrm{Det} H^2(X;\mathbb{R})| \otimes |\mathrm{Det} H^3(X;\mathbb{R})^*| \end{aligned} \tag{7.22}$$

It is defined by the expression [RS2]:

$$T^a_X = \delta_{|\mathrm{Det} H^\bullet(X;\mathbb{R})|} \cdot \exp\Big[\frac{1}{2} \sum_{q=0}^{\dim X} (-1)^q q \zeta_q'(0)\Big] \tag{7.23}$$

The inner product on the space $\mathcal{H}^q(X)$ of harmonic q-forms defined by the product (7.15) on forms determines an inner product on $H^q(X;\mathbb{R}) \cong \mathcal{H}^q(X)$ and thus, a density $\delta_{|\mathrm{Det} H^\bullet(X;\mathbb{R})|}$ on $|\mathrm{Det} H^\bullet(X;\mathbb{R})|$. If $b_q = \dim \mathcal{H}^q(X)$ and $\nu_1^{(q)}, \dots, \nu_{b_q}^{(q)}$ is any orthonormal basis for $\mathcal{H}^q(X)$, then

$$\delta_{|\mathrm{Det} H^\bullet(X;\mathbb{R})|} \;=\; \bigotimes_{q=0}^{\dim X} |\nu^{(q)}|^{(-1)^q}\,, \tag{7.24}$$

where $\nu^{(q)} = \nu_1^{(q)} \wedge \cdots \wedge \nu_{b_q}^{(q)}$. According to the definition of regularized determinants we have $\det{}'\,\Delta_q = \mathrm{e}^{-\zeta_q'(0)}$. Thus the Ray-Singer torsion of the closed connected 3-manifold X can be represented as

$$\begin{aligned} T^a_X \;=\; &\big[(\det{}'\Delta_0)^0 (\det{}'\Delta_1)^1 (\det{}'\Delta_2)^{-2} (\det{}'\Delta_3)^3\big]^{\frac{1}{2}} \times \\ &\times |\nu^{(0)}| \otimes |\nu^{(1)}|^{-1} \otimes |\nu^{(2)}| \otimes |\nu^{(3)}|^{-1} \end{aligned} \tag{7.25}$$

The Hodge $\star$-operator determines isomorphisms $\Delta_q \cong \Delta_{3-q}$ and by Poincaré duality $H^q(X;\mathbb{R}) \cong H^{3-q}(X;\mathbb{R})^*$. Moreover, any orthonormal basis $\nu^{(0)}$ of $\mathcal{H}^0(X) \cong \mathbb{R}$

is a constant such that $|\nu^{(0)}| = \big[\mathrm{Vol} X\big]^{-\frac{1}{2}}$. Hence the square root of the analytic torsion T_X^a can be regarded as a density $(T_X^a)^{\frac{1}{2}} \in |\mathrm{Det} H^1(X;\mathbb{R})^*|$ which is given by the expression

$$(T_X^a)^{\frac{1}{2}} \;=\; \frac{1}{\big[\mathrm{Vol} X\big]^{\frac{1}{2}}}\, \frac{\big[\det{}'\,\Delta_0\big]^{\frac{3}{4}}}{\big[\det{}'\,\Delta_1\big]^{\frac{1}{4}}}\, \nu\,, \tag{7.26}$$

where $\nu = |\nu^{(1)}|^{-1} = |\nu_1^{(1)} \wedge \cdots \wedge \nu_{b_1}^{(1)}|^{-1}$ for any orthonormal basis $\nu_1^{(1)}, \cdots, \nu_{b_1}^{(1)}$ of $\mathcal{H}^1(X)$. In view of the isomorphism $\mathcal{M}_P \cong H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$, the square root $(T_X^a)^{\frac{1}{2}}$ of the analytic torsion defines an invariant density on the moduli space of flat connections $\mathcal{M}_P$. Using (7.26), the expression (7.21) reads:

$$Z_{X,p} \;=\; k^{m_X}\, \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta_P)}\, \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\eta(-\star d)} \int\limits_{\mathcal{M}_P} (T_X^a)^{\frac{1}{2}} \tag{7.27}$$

It is proved in [RS2] that the analytic torsion of a closed manifold is independent of the metric. Thus T_X^a is a manifold invariant. The only metric dependence in (7.27) comes from the phase factor $\mathrm{e}^{\frac{\pi \mathrm{i}}{4}\eta(-\star d)}$. Since, after all, *we* choose *how* to define $Z_{X,p}$, we are going to modify our original definition (7.17) by multiplying that expression with $\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}\eta(-\star d)}$. This makes the resulting expression metric independent. We remark that in the non-abelian Chern-Simons theory [Wi1], Witten compensates the metric dependence of the path integral by adding to the original action a counterterm, the gravitational Chern-Simons action.

Starting with a heuristic functional integral formula, we have thus succeeded to associate to the closed connected 3-manifold X the topological invariant

$$Z_X \;=\; k^{m_X} \sum_{p \in \mathrm{Tors} H^2(X;\mathbb{Z})} \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(\Theta_P)} \int\limits_{\mathcal{M}_P} (T_X^a)^{\frac{1}{2}} \tag{7.28}$$

We note that the factor k^{m_X} agrees with the one appearing in [FG], in the analogous expression of the closed 3-manifold invariant for the non-abelian version of the Chern-Simons theory.

We treat now the case of a compact connected oriented 3-manifold X *with boundary* ∂X. We fix a trivializable $\mathbb{T}$-bundle Q over ∂X and a section $s : \partial X \to Q$. Then, for each $p \in \mathrm{Tors} H^2(X;\mathbb{Z})$, we choose a $\mathbb{T}$-bundle $P \to X$ with $c_1(P) = p$

and fix a bundle isomorphism $\phi_P : \partial P \to Q$. The section $s : \partial X \to Q$ determines a section $s_P = \phi_P^{-1} \circ s : \partial X \to \partial P$. If η is a connection on Q, then we define the space

$$\mathcal{A}_P(\eta) \;=\; \{\Theta \in \mathcal{A}_P \mid \partial\Theta = \phi_P^* \eta\}$$

of connections on P whose restriction $\partial\Theta = \Theta\big|_{\partial X}$ to the boundary of X is identified with η under the given bundle isomorphism over ∂X. The space $\mathcal{A}_P(\eta)$ is an affine space with vector space $2\pi\mathrm{i}\Omega^1_{1,tan}(X;\mathbb{R})$, where $\Omega^1_{1,tan}(X;\mathbb{R})$ denotes the space of 1-forms A on X whose geometrical restriction to ∂X is zero (i.e. the pullback $i^*A = 0$ under the inclusion map $i : \partial X \to X$).

Let $\mathcal{G}_P(e) \cong \mathcal{G}_X(e)$ denote the subgroup in $\mathcal{G}_P \cong \mathcal{G}_X$ of gauge transformations which restrict to the identity map over ∂X. The group $\mathcal{G}_P(e)$ acts on $\mathcal{A}_P(\eta)$ through (2.1) and the action is free. Given the section $s_P : \partial X \to \partial P$, the Chern-Simons functional $S_{X,P}(s_P, \cdot) : \mathcal{A}_P(\eta) \to \mathbb{R}/\mathbb{Z}$ defined by (3.9) is invariant under the $\mathcal{G}_P(e)$-action on $\mathcal{A}_P(\eta)$. As for the closed manifold case previously discussed, we set

$$Z_{X,p}(\eta) \;=\; \int\limits_{\mathcal{A}_P(\eta)/\mathcal{G}_P(e)} [\mathcal{D}\Theta]\, \mathrm{e}^{\pi\mathrm{i}kS_{X,P}(s_P,\Theta)} \tag{7.29}$$

and using similar methods aim to show in what follows that one can make sense of such a functional integral.

For this purpose let us choose a Riemannian metric on X. The inner product defined on the tangent space by the metric determines, at each point of ∂X, a normal vector to the boundary. Therefore, a differential form α on X can be decomposed into a tangential and a normal component $\alpha = \alpha_{tan} + \alpha_{norm}$ and one has $(\star\alpha)_{norm} = \star(\alpha_{tan})$, $(d\alpha)_{tan} = d(\alpha_{tan})$, $(d^*\alpha)_{norm} = d^*(\alpha_{norm})$. We introduce the subspaces in $\Omega^q(X;\mathbb{R})$ of q-forms on X satisfying *relative* boundary conditions [RS1]:

$$\Omega^q_{tan}(X;\mathbb{R}) = \{\alpha \in \Omega^q(X;\mathbb{R}) \mid \alpha_{tan} = (d^*\alpha)_{tan} = 0 \;\text{ on } \partial X\}\,,$$

and *absolute* boundary conditions:

$$\Omega^q_{norm}(X;\mathbb{R}) = \{\beta \in \Omega^q(X;\mathbb{R}) \mid \beta_{norm} = (d\beta)_{norm} = 0 \text{ on } \partial X\}.$$

Then we let Δ^{tan}_q denote the Laplace operator $d^*d + dd^*$ acting on $\Omega^q_{tan}(X;\mathbb{R})$ and Δ^{norm}_q the Laplace operator on $\Omega^q_{norm}(X;\mathbb{R})$. If i^* is the pullback under the inclusion $i : \partial X \to X$, the conditions $i^*\alpha = 0$ and $i^*(\star\beta) = 0$ are equivalent to the conditions $\alpha_{tan} = 0$ on ∂X and $\beta_{norm} = 0$ on ∂X, respectively. The Hodge-deRham theory for manifolds with boundary gives the isomorphisms [RS1, Mü]

$$\begin{aligned} H^q(X,\partial X;\mathbb{R}) &\cong \mathcal{H}^q_{tan}(X) \\ H^q(X;\mathbb{R}) &\cong \mathcal{H}^q_{norm}(X), \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}^q_{tan}(X) &= \{\alpha \in \Omega^q(X;\mathbb{R}) \mid \alpha_{tan} = 0 \text{ on } \partial X,\ d\alpha = d^*\alpha = 0 \text{ on } X\} \\ \mathcal{H}^q_{norm}(X) &= \{\beta \in \Omega^q(X;\mathbb{R}) \mid \beta_{norm} = 0 \text{ on } \partial X,\ d\beta = d^*\beta = 0 \text{ on } X\} \end{aligned}$$

are the spaces of harmonic forms on X with relative and absolute boundary conditions.

Let $\hat{\mu}$ be the measure on $\mathcal{A}_P(\eta)$ determined by the inner product (7.15) defined on each tangent space $T_\Theta\mathcal{A}_P(\eta) \cong \Omega^1_{1,tan}(X;\mathbb{R})$ by the metric on X and let μ be the induced measure on the quotient space $\mathcal{A}_P(\eta)/\mathcal{G}_P(e)$. For each $\Theta \in \mathcal{A}_P(\eta)$, we have the linear map $\tau_\Theta = d : \operatorname{Lie}\mathcal{G}_P(e) \cong \Omega^0_{tan}(X;\mathbb{R}) \to T_\Theta\mathcal{A}_P(\eta) \cong \Omega^1_{1,tan}(X;\mathbb{R})$ determined by the $\mathcal{G}_P(e)$-action on $\mathcal{A}_P(\eta)$.

The stationary points of the Chern-Simons functional $S_{X,P}(s_P,\cdot) : \mathcal{A}_P(\eta) \to \mathbb{R}/\mathbb{Z}$ form the subspace of flat connections $\mathcal{A}^f_P(\eta) = \{\Theta \in \mathcal{A}_P(\eta) \mid F_\Theta = d\Theta = 0\}$. We assume that the connection η over ∂X is flat and extends to flat connections over X. Otherwise the spaces $\mathcal{A}^f_P(\eta)$ are empty. Let us fix an arbitrary flat connection $\Theta_P \in \mathcal{A}^f_P(\eta)$. Then any connection $\Theta \in \mathcal{A}_P(\eta)$ can be expressed as $\Theta = \Theta_P + 2\pi\mathrm{i}A$, for some $A \in \Omega^1_{1,tan}(X;\mathbb{R})$, and it follows from (3.18) that

$$S_{X,P}(s_P,\Theta) = S_{X,P}(s_P,\Theta_P) - (A,\star dA) \pmod 1$$

Proceeding in the same way as for the closed manifold case, we mimic the finite dimensional model of Remark (7.1) and *formally* define the partition function of the quadratic functional $S_{X,P}(s_P,\cdot)$ as

$$Z_{X,p}(\eta) = \frac{1}{\operatorname{Vol}\mathcal{G}_P(e)} \int\limits_{\mathcal{A}_P(\eta)} \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(s_P,\Theta)}\,\hat{\mu} \tag{7.30}$$

$$= \int\limits_{\mathcal{A}_P(\eta)/\mathcal{G}_P(e)} \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(s_P,\Theta)} \big[\det{}'(\tau_\Theta^*\tau_\Theta)\big]^{\frac{1}{2}}\,\mu \tag{7.30 $'$}$$

$$= \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(s_P,\Theta_P)} \int\limits_{\mathcal{M}_P(\eta)} \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\operatorname{sgn}(-\star d)} \frac{\big[\det{}'(\tau_\Theta^*\tau_\Theta)\big]^{\frac{1}{2}}}{\big[\det{}'(-k\star d)\big]^{\frac{1}{2}}}\,\nu_t \tag{7.30 $''$}$$

The deRham theory for manifolds with boundary gives the following identification $H^1(X,\partial X;\mathbb{R}) = (\operatorname{Ker} d\big|_{\Omega^1_{1,tan}(X;\mathbb{R})})/(\operatorname{Im} d\big|_{\Omega^0_{tan}(X;\mathbb{R})})$. Thus the quotient space $\mathcal{M}_P(\eta) = \mathcal{A}^f_P(\eta)/\mathcal{G}_P(e)$ is isomorphic to the torus $H^1(X,\partial X;\mathbb{R})/H^1(X,\partial X;\mathbb{Z})$. The measure ν_t on $\mathcal{M}_P(\eta)$ appearing in (7.30$''$) is the measure induced by the inner product determined on $H^1(X,\partial X;\mathbb{R})$ through the isomorphism with $\mathcal{H}^1_{tan}(X)$. The expression (7.30$'$) will be taken as the formal definition for $Z_{X,p}(\eta)$ in (7.29). We are going to show in what follows that (7.30$''$) has rigorous mathematical meaning if one takes the regularized determinants and signatures of the operators therein.

To evaluate $\det'(-k\star d)$ we are going to use the results in Remark (7.9). Let $S = -k\star d$ acting on the space $\Omega^1_{tan}(X;\mathbb{R})$ and $\tilde{S} = \left(\begin{smallmatrix} 0 & S \\ S^* & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -k\star d \\ -k\star d & 0\end{smallmatrix}\right)$ acting on $\Omega^1_{tan}(X;\mathbb{R}) \oplus \Omega^1_{norm}(X;\mathbb{R})$. We also let $\tilde{T}$ be the operator $\tilde{T} = \left(\begin{smallmatrix} kd & 0 \\ 0 & kd\end{smallmatrix}\right)$ on the space $\Omega^0_{tan}(X;\mathbb{R}) \oplus \Omega^0_{norm}(X;\mathbb{R})$. Then we find that $\tilde{S}^2 + \tilde{T}\tilde{T}^* = k^2(\Delta_1^{tan} \oplus \Delta_1^{norm})$ and $\tilde{T}^*\tilde{T} = k^2(\Delta_0^{tan} \oplus \Delta_0^{norm})$. According to the definition (7.8), we have $\det' S =$

$|\det'(S^*S)|^{\frac{1}{2}} = |\det' \tilde{S}|^{\frac{1}{2}}$. Using this and the relation (7.10), we can write

$$\begin{aligned}
\det{}'(-k \star d) &= |\det{}'\tilde{S}|^{\frac{1}{2}} = \frac{\left|\det{}'(\tilde{S}^2 + \tilde{T}\tilde{T}^*)\right|^{\frac{1}{4}}}{\left|\det{}'(\tilde{T}^*\tilde{T})\right|^{\frac{1}{4}}} \\
&= \frac{\left|\det{}'(k^2\Delta_1^{tan})\ \det{}'(k^2\Delta_1^{norm})\right|^{\frac{1}{4}}}{\left|\det{}'(k^2\Delta_0^{tan})\ \det{}'(k^2\Delta_0^{norm})\right|^{\frac{1}{4}}} \\
&= \frac{k^{-\frac{1}{2}[\dim H^1(X;\mathbb{R}) + \dim H^1(X,\partial X;\mathbb{R})]}}{k^{-\frac{1}{2}[\dim H^0(X;\mathbb{R}) + \dim H^0(X,\partial X;\mathbb{R})]}} \frac{\left|\det{}'\,\Delta_1^{tan}\ \det{}'\,\Delta_1^{norm}\right|^{\frac{1}{4}}}{\left|\det{}'\,\Delta_0^{tan}\ \det{}'\,\Delta_0^{norm}\right|^{\frac{1}{4}}}
\end{aligned}$$

Inserting the above result together with $\det'(\tau_\Theta^*\tau_\Theta) = \det'\,\Delta_0^{tan}$ into (7.30″), we obtain

$$Z_{X,p}(\eta) = k^{m_X}\, \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(s_P,\Theta_P)}\, \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\eta(-\star d)} \times \\ \times \int_{\mathcal{M}_P(\eta)} \frac{\left|\det{}'\,\Delta_0^{tan}\right|^{\frac{5}{8}} \left|\det{}'\,\Delta_0^{norm}\right|^{\frac{1}{8}}}{\left|\det{}'\,\Delta_1^{tan}\right|^{\frac{1}{8}} \left|\det{}'\,\Delta_1^{norm}\right|^{\frac{1}{8}}}\, \nu_t \tag{7.31}$$

with m_X defined as in (5.16). We are going to relate the term inside the integral sign in the expression (7.31) to the Ray-Singer analytic torsion of X.

The same definition (7.23) of the Ray-Singer analytic torsion as a density T_X^a on $|\mathrm{Det} H^\bullet(X;\mathbb{R})|$ applies when X is a Riemannian manifold with boundary [Mü, V]. In this case the zeta-function ζ_q in (7.23) is that of the Laplace operator Δ_q^{norm} on $\Omega^q_{norm}(X;\mathbb{R})$ and the density $\delta_{|\mathrm{Det} H^\bullet(X;\mathbb{R})|}$ on $|\mathrm{Det} H^\bullet(X;\mathbb{R})|$ corresponds to the natural inner product on $H^\bullet(X;\mathbb{R}) \cong \mathcal{H}^\bullet_{norm}(X)$. Thus the Ray-Singer analytic torsion of the compact connected 3-manifold X with nonempty boundary ∂X is the density $T_X^a \in |\mathrm{Det} H^0(X;\mathbb{R})| \otimes |\mathrm{Det} H^1(X;\mathbb{R})^*| \otimes |\mathrm{Det} H^2(X;\mathbb{R})|$ given by

$$\begin{aligned}
T_X^a &= \left[(\det{}'\Delta_0^{norm})^0 (\det{}'\Delta_1^{norm})^1 (\det{}'\Delta_2^{norm})^{-2} (\det{}'\Delta_3^{norm})^3\right]^{\frac{1}{2}} \times \\
&\quad \times |\nu^{(0)}_{norm}| \otimes |\nu^{(1)}_{norm}|^{-1} \otimes |\nu^{(2)}_{norm}|
\end{aligned}$$

where $\nu^{(q)}_{norm} = \nu^{(q)}_1 \wedge \cdots \wedge \nu^{(q)}_{b_q}$, with $b_q = \dim \mathcal{H}^q_{norm}(X)$ and $\nu^{(q)}_1, \ldots, \nu^{(q)}_{b_q}$ an orthonormal basis for $\mathcal{H}^q_{norm}(X)$. An orthonormal basis $\nu^{(0)}_{norm}$ of $\mathcal{H}^0(X) \cong \mathbb{R}$ is such that $|\nu^{(0)}_{norm}| = \left[\mathrm{Vol} X\right]^{-\frac{1}{2}}$. Poincaré duality gives the isomorphisms $H^q(X;\mathbb{R}) \cong H^{3-q}(X,\partial X;\mathbb{R})^*$. Also, the spaces $\Omega^q_{norm}(X;\mathbb{R})$ and $\Omega^{3-q}_{tan}(X;\mathbb{R})$ are isomorphic

under the Hodge $\star$-operator and $\det' \Delta_q^{norm} = \det' \Delta_{3-q}^{tan}$. In consequence, the square root of the analytic torsion can be regarded as a half-density $(T_X^a)^{\frac{1}{2}}$ in $|\mathrm{Det} H^1(X;\mathbb{R})^*|^{\frac{1}{2}} \otimes |\mathrm{Det} H^1(X,\partial X;\mathbb{R})^*|^{\frac{1}{2}}$ which is given by the expression

$$(T_X^a)^{\frac{1}{2}} = \frac{1}{\big[\mathrm{Vol} X\big]^{\frac{1}{4}}} \frac{\big[\det' \Delta_0^{tan}\big]^{\frac{3}{4}} \big[\det' \Delta_1^{norm}\big]^{\frac{1}{4}}}{\big[\det' \Delta_1^{tan}\big]^{\frac{1}{2}}} (\nu_n)^{\frac{1}{2}} \otimes (\nu_t)^{\frac{1}{2}} \tag{7.32}$$

Here $\nu_n = |\nu_{norm}^{(1)}|^{-1} = |\nu_1^{(1)} \wedge \cdots \wedge \nu_{b_1}^{(1)}|^{-1}$, for any orthonormal basis $\nu_1^{(1)}, \ldots, \nu_{b_1}^{(1)}$ of harmonic forms of $\mathcal{H}_{norm}^1(X)$. Similarly, $\nu_t = |\nu_{tan}^{(1)}|^{-1} = |\tilde{\nu}_1^{(1)} \wedge \cdots \wedge \tilde{\nu}_{l_1}^{(1)}|^{-1}$, for any orthonormal basis $\tilde{\nu}_1^{(1)}, \ldots, \tilde{\nu}_{l_1}^{(1)}$ of $\mathcal{H}_{tan}^1(X)$, where $l_1 = \dim \mathcal{H}_{tan}^1(X)$. Having in view of the isomorphism $\mathcal{M}_P(\eta) \cong H^1(X,\partial X;\mathbb{R})/H^1(X,\partial X;\mathbb{Z})$, we note that

$$\int\limits_{\mathcal{M}_P(\eta)} (T_X^a)^{\frac{1}{2}} \equiv \Big[\int\limits_{\mathcal{M}_P(\eta)} \nu_t \Big] (T_X^a)^{\frac{1}{2}} \otimes (\nu_t)^{-1} \tag{7.33}$$

is a half-density in $|\mathrm{Det} H^1(X;\mathbb{R})^*|^{\frac{1}{2}} \otimes |\mathrm{Det} H^1(X,\partial X;\mathbb{R})|^{\frac{1}{2}}$.

As in the case of a closed manifold, it was proven in [V] that the analytic torsion T_X^a of a manifold X with boundary is independent of the metric chosen on X (the metric is assumed to be a direct product metric near ∂X). On the other hand the partition function $Z_{X,P}(\eta)$ in (7.31) is a metric dependent complex number. In the light of the previous observations on the analytic torsion, we remark that we can get rid of this metric dependence if we multiply $Z_{X,P}(\eta)$ by the half-density $\Upsilon \in |\mathrm{Det} H^1(X;\mathbb{R})^*|^{\frac{1}{2}} \otimes |\mathrm{Det} H^1(X,\partial X;\mathbb{R})|^{\frac{1}{2}}$ defined by

$$\Upsilon = 2^{-\frac{\chi(\partial X)}{2}} \frac{\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}\eta(-\star d)}}{\big[\mathrm{Vol} X\big]^{\frac{1}{4}}} \frac{\big[\det' \Delta_0^{tan}\big]^{\frac{1}{8}} \big[\det' \Delta_0^{norm}\big]^{-\frac{1}{8}}}{\big[\det' \Delta_1^{tan}\big]^{\frac{3}{8}} \big[\det' \Delta_1^{norm}\big]^{-\frac{3}{8}}} (\nu_n)^{\frac{1}{2}} \otimes (\nu_t)^{-\frac{1}{2}}, \tag{7.34}$$

where $\chi(\partial X)$ is the Euler characteristic of ∂X. Thus, multiplying (7.31) by (7.34) and having (7.33) in view, we obtain

$$Z_{X,p}(\eta)\, \Upsilon = k^{m_X}\, \mathrm{e}^{\pi \mathrm{i} k S_{X,P}(s_P,\Theta_P)} \int\limits_{\mathcal{M}_P(\eta)} 2^{-\frac{\chi(\partial X)}{2}} (T_X^a)^{\frac{1}{2}}$$

which is independent of the metric on X.

Hence to a compact connected 3-manifold X with boundary ∂X the path integral approach outlined above associates, for every flat connection η over ∂X

extending to flat connections over X, the half-density

$$Z_X(\eta) \in |\mathrm{Det} H^1(X;\mathbb{R})^*|^{\frac{1}{2}} \otimes |\mathrm{Det} H^1(X,\partial X;\mathbb{R})|^{\frac{1}{2}}$$

given by the expression

$$\begin{aligned} Z_X(\eta) &= \sum_{p\in \mathrm{Tors} H^2(X;\mathbb{Z})} Z_{X,p}(\eta)\,\Upsilon \\ &= k^{m_X} \sum_{p\in \mathrm{Tors} H^2(X;\mathbb{Z})} \mathrm{e}^{\pi i k S_{X,P}(s_P,\Theta_P)} \int_{\mathcal{M}_P(\eta)} 2^{-\frac{\chi(\partial X)}{2}} (T^a_X)^{\frac{1}{2}} \end{aligned} \tag{7.35}$$

According to the remarks made in Sect.2 regarding the group $H^1(X,\partial X;\mathbb{T})$, we have $\pi_0(H^1(X,\partial X;\mathbb{T})) \cong \mathrm{Tors} H^2(X,\partial X;\mathbb{Z}) \cong \mathrm{Tors} H^2(X;\mathbb{Z})$ and each component of $H^1(X,\partial X;\mathbb{T})$ is isomorphic to the torus $H^1(X,\partial X;\mathbb{R})/H^1(X,\partial X;\mathbb{Z})$. Since

$$\begin{aligned} \int_{\mathcal{M}_P(\eta)} (T^a_X)^{\frac{1}{2}} &= \int_{H^1(X,\partial X;\mathbb{R})/H^1(X,\partial X;\mathbb{Z})} (T^a_X)^{\frac{1}{2}} \\ &= \frac{1}{[\#\pi_0(H^1(X,\partial X;\mathbb{T}))]} \int_{H^1(X,\partial X;\mathbb{T})} (T^a_X)^{\frac{1}{2}}\,, \end{aligned}$$

for all P, we can rewrite (7.35) as

$$Z_X(\eta) = \frac{k^{m_X}}{[\#\mathrm{Tors} H^2(X;\mathbb{Z})]} \sum_{p\in \mathrm{Tors} H^2(X;\mathbb{Z})} \mathrm{e}^{\pi i k S_{X,P}(s_P,\Theta_P)} \int_{H^1(X,\partial X;\mathbb{T})} 2^{-\frac{\chi(\partial X)}{2}} (T^a_X)^{\frac{1}{2}} \tag{7.36}$$

Ray and Singer conjectured [RS1] that for a closed manifold X the analytic torsion norm T^a_X and the Reidemeister torsion norm T_X on $|\mathrm{Det} H^\bullet(X;\mathbb{R})|$ coincide

$$T^a_X = T_X. \tag{7.37}$$

A proof of this fact can be found in [V]. Thus, in view of (7.37), the expression (7.28) for the invariant Z_X associated to the closed connected oriented 3-manifold X through the path integral approach coincides with the expression (5.17) from the geometric quantization approach.

For a compact manifold X with nonempty boundary ∂X, [V] proves that the two norms on $|\mathrm{Det} H^\bullet(X;\mathbb{R})|$, the analytic torsion norm T_X^a and the Reidemeister torsion norm T_X, are related by

$$T_X^a = 2^{\frac{\chi(\partial X)}{2}} T_X. \tag{7.38}$$

Referring to the definition of the Chern-Simons section in Sect.4 and to the discussion in Sect.5, we note that the terms $e^{\pi i k S_{X,P}(s_P,\Theta_P)}$ in (7.36) give rise to sections of the prequantum line bundle $\mathcal{L}_{\partial X}$ over the Lagrangian image $\Lambda_X \subset \mathcal{M}_{\partial X}$ of the moduli space of flat $\mathbb{T}$-connections $\mathcal{M}_X$. In view of (7.33) and of the isomorphism (5.12), the half-density $\int\limits_{H^1(X,\partial X;\mathbb{T})} (T_X^a)^{\frac{1}{2}}$ can be interpreted as a section of the bundle of half-densities on Λ_X. Hence, with the identification (7.38), the expression (7.36) is seen to correspond to the vector Z_X in (5.15). It is interesting that the pure path integral (7.31) and the half-density factor (7.34) introduced to compensate the metric dependence of (7.31) combine together to produce a result which agrees with the one obtained in the geometric quantization approach.

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